HARMONIC MAPS OF CONIC SURFACES WITH CONE ANGLES LESS THAN 2π

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Abstract. We prove the existence and uniqueness of harmonic maps in degree one homotopy classes of closed, orientable surfaces of positive genus, where the target has non-positive gauss curvature and conic points with cone angles less than 2π. For a homeomorphism w of such a surface, we prove existence and uniqueness of minimizers in the homotopy class of w relative to the inverse images of the cone points with cone angles less than or equal to π. We show that such maps are homeomorphisms and that they depend smoothly on the target metric. For fixed geometric data, the space of minimizers in relative degree one homotopy classes is a complex manifold of (complex) dimension equal to the number of cone points with cone angles less than or equal to π.

When the genus is zero, we prove the same relative minimization provided there are at least three cone points of cone angle less than or equal to π.

1. Introduction

The study of harmonic maps into singular spaces, initiated in [GS], has reached a refined state; beyond general existence and uniqueness theorems, there are regularity and compactness results in the presence of minimal regularity assumptions on the spaces involved, see, among many others, [KS], [Mese1], [DM], [W2], and [EF].

Particularly detailed results are available in the case of maps of surfaces. In [K], Kuwert studied degree one harmonic maps of closed Riemann surfaces into flat, conic surfaces with cone angles bigger than 2π. He showed that the minimizing maps can be obtained as limits of diffeomorphisms, and that the inverse image under a degree one harmonic map of each point in the singular set is the union of a finite number of vertical arcs of the Hopf differential. Away from this inverse image the map is a local diffeomorphism. In [Mese1], Mese proved the same when the target is a metric space with bounded curvature, in particular when it is a conic surface with cone angles bigger than 2π.

In this paper, we study two main topics: energy minimizing maps from Riemann surfaces into conic surfaces with cone angles less than 2π, and energy minimizing maps from *punctured* Riemann surfaces into conic surfaces with cone angles less than or equal to π. We state our results somewhat sketchily at first, assuming for simplicity's sake that there is just one cone point. The energy functional is conformally invariant with respect to the domain metric (see §3.1), so we state all results in terms of conformal structures, $\mathfrak{c}$, on the domain. First, we have

Theorem 1. *Let Σ be a closed surface of genus > 0, equipped with a conformal structure $\mathfrak{c}$ and a Riemannian metric G, with a conic point p of cone angle less than 2π and non-positive Gauss curvature away from p. Let $\phi : \Sigma \longrightarrow \Sigma$ be a homeomorphism. Then there is a unique*

map $u : \Sigma \longrightarrow \Sigma$ which minimizes energy in the homotopy class of ϕ. This map satisfies

$$u^{-1}(p) \text{ is a single point}$$

and $u : \Sigma - u^{-1}(p) \longrightarrow \Sigma - p$ is a diffeomorphism.

That $u^{-1}(p)$ is a single point is consistent with the work of Hardt and Lin on maps into round cones in Euclidean space, see [HL]. Second, we have

Theorem 2. *Let $\Sigma, \mathfrak{c}$, and G be as in the previous theorem. If the cone angle at p is less than or equal to π, then for each $q \in \Sigma$ and each homeomorphism ϕ of Σ with $\phi(q) = p$, there is a unique map $u : \Sigma \longrightarrow \Sigma$ with $u(q) = p$ which minimizes energy in the rel. q homotopy class of ϕ (see (2.29) for the definition of relative homotopy class). This map satisfies*

$$u^{-1}(p) = q$$

and $u : \Sigma - q \longrightarrow \Sigma - p$ is a diffeomorphism.

This problem is motivated by the role of harmonic maps in Teichmüller theory and recent work that extends classical uniformization results to the case of conic metrics on punctured surfaces. The uniformization theorems for cone metrics of McOwen and Troyanov, [Mc1], [Mc2], [Tro], recent work by Schumacher and Trapani, [ST], [ST2], and unpublished work of Mazzeo and Weiss has shown that the role the hyperbolic geometry plays in Teichmüller theory is played by conic metrics in the punctured case. In the unpunctured case, harmonic maps enter the picture in the works of [Tr], [W1], [W2], and [W3], in the creation of various functionals and in the important parametrization of Teichmüller space by holomorphic quadratic differentials. This paper lays the groundwork for the extension of these results to the punctured case if the uniformizing metrics are conic, and makes available — in case the cone angles are less than π — harmonic maps of punctured Riemann surfaces.

The proofs follow the method of continuity, and accordingly the paper is divided into a portion with a perturbation result and a portion with a convergence result. In both we make frequent use of the fact that minimizers as in both theorems solve a differential equation (see (2.4)). To be precise, let $\mathcal{M}_{conic}(p, \alpha)$ denote the space of smooth metrics on $\Sigma - p$ with a cone point at p of cone angle $2\pi\alpha$ (see §2.2). The 'tension field' operator τ is a second order, quasi-linear, elliptic partial differential operator, arising as the Euler-Lagrange equation for the energy functional, which takes a triple (u, g, G) of maps and metrics to a vector field over u, denoted by $\tau(u, g, G)$. For $g \in \mathcal{M}_{conic}(q, \alpha)$, $G \in \mathcal{M}_{conic}(p, \alpha)$, in section 5 we show that a diffeomorphism $u : (\Sigma - q, g) \longrightarrow (\Sigma - p, G)$ subject to certain conditions on the behavior near q minimizes energy in its rel. q homotopy class if $\tau(u, g, G) = 0$.

The perturbation result is an application of the Implicit Function Theorem to the tension field operator. Fix $g \in \mathcal{M}_{conic}(q, \alpha)$, $G \in \mathcal{M}_{conic}(p, \alpha)$ and a minimizer u as in Theorem 2. The proofs rest mainly in finding the right space of perturbations of u, call them $\mathcal{P}(u)$, and the right space of perturbations of g, call them $\mathcal{M}^*(g) \subset \mathcal{M}_{conic}(q, \alpha)$, so that τ acting on $\mathcal{P}(u) \times \mathcal{M}^*(g)$ has non-degenerate differential in the $\mathcal{P}(u)$ direction at (u, g). There are two not-quite-correct points in this last sentence. First, to apply the Implicit Function Theorem, one needs to work with Banach and not Fréchet manifolds like $\mathcal{M}_{conic}(p, \alpha)$; for a precise definition of the spaces we use see section 3. Second, the map τ actually takes values in a bundle, specifically the bundle $\mathsf{E} \longrightarrow \mathcal{P}(u) \times \mathcal{M}^*(g)$ whose fiber over $(\widetilde{u}, \widetilde{g})$ is (some Banach space of) sections of $\widetilde{u}^*\mathrm{T}\Sigma$, so in fact we do not show that τ has surjective differential but rather that τ is transversal to the zero section of E.

If z denotes conformal coordinates near q, the linear operator L can be written $L = |z|^{2\alpha}\,\widetilde{L}$, where $\widetilde{L}$ falls into a broad class of linear operators known as elliptic b-differential operators, pioneered and elaborated by R. Melrose, and used subsequently in countless settings (§6.2). For detailed properties of b-operators, see [Me] and [Me-Me]. The main difficulty we encounter is that L is degenerate on a natural space of perturbations. Following the example of previous authors, including [MP], we supplement the domain of τ with 'geometric' perturbations; in our case, we add a space $\mathcal{C}$ of diffeomorphisms of the domain which look like conformal dilations, rotations, and translations near the cone point (§3).

As we will discuss in section 6.2, the natural domains for L are weighted Banach spaces $r^c\mathcal{X}_b^{2,\gamma}$, which for the moment should be thought of as vector fields vanishing to order r^c near q. In particular L acts from $r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}$ to $r^{1-\epsilon-2\alpha}\mathcal{X}_b^{2,\gamma}$, and is Fredholm for sufficiently small $\epsilon > 0$. Let $\mathcal{K}$ denote its cokernel (see (6.3)). The leading order behavior of a vector field in $\mathcal{K}$ near the inverse image of the cone point is characterized by the homogeneous solutions of a related 'indicial' equation, c.f. Lemma 6.8. A dichotomy between the behavior of elements in $\mathcal{K}$ near cone points of cone angle less than π and cone angle greater than π arises.

Lemma 1.1. *Let $\psi \in \mathcal{K}$, and suppose that G has only one cone point, p, with $u^{-1}(p) = q$. If the cone angle $2\pi\alpha$ is bigger than π, then near q*

$$\psi(z) = w + \frac{\overline{a}}{1-\alpha}\,|z|^{2(1-\alpha)} + \mathcal{O}(|z|^{2(1-\alpha)+\delta}) \tag{1.1}$$

for some $w, a \in \mathbb{C}$ and $\delta > 0$. If the cone angle is less than π, then near q

$$\psi(z) = \mu z + \mathcal{O}(|z|^{1+\delta}) \tag{1.2}$$

for some $\mu \in \mathbb{C}$ and $\delta > 0$.

This lemma is central to the proof of the main theorems, since an accurate appraisal of the cokernel is needed to show that the geometric perturbations described above are sufficient to give a surjective problem.

To prove energy minimization, both in relative and absolute homotopy classes, we use an argument from [CH], in which the authors prove uniqueness for harmonic maps of surfaces with genus bigger than 1. They show that the pullback of the target metric via a harmonic map can be written as a sum of a two metrics, one conformal to the domain metric and the other with negative curvature, thereby decomposing the energy functional into a sum of two functionals which the harmonic map jointly minimizes. In section 5, we adapt this argument to prove uniqueness in the conic setting.

For fixed domain and target structures $\mathfrak{c}$ and G, if the cone point of G has cone angle less than π, the harmonic maps in Theorem 2 from $(\Sigma, \mathfrak{c})$ to (Σ, G) are parametrized by points q and rel. q homotopy classes of diffeomorphisms taking q to p. Denote this space by $\mathcal{H}arm_{\mathfrak{c},G}$. Given any diffeomorphism ϕ of Σ, let $[\phi]_{rel.} = [\phi; \phi^{-1}(p)]$, and denote the corresponding element of $\mathcal{H}arm_{\mathfrak{c},G}$ by $u_{[\phi]_{rel.}}$. One of our main results is a formula for the gradient of the energy functional on $\mathcal{H}arm_{\mathfrak{c},G}$ in terms of the Hopf differential $\Phi(u_{[\phi]_{rel.}})$ of $u_{[\phi]_{rel.}}$ (see section 5.1 for a definition of the Hopf differential). It turns out that $\Phi(u_{[\phi]_{rel.}})$ is holomorphic on $\Sigma - q$ with at most a simple pole at q. Given a path u_t in $\mathcal{H}arm_{\mathfrak{c},G}$ with

$u_0 = u_{[\phi]_{rel.}}$, and writing $J := \frac{d}{dt}\big|_{t=0} u_t$, the gradient is given by

$$\frac{d}{dt}\bigg|_{t=0} E(u_t) = \Re 2\pi i \operatorname{Res}|_q \iota_J \Phi(u_{[\phi]_{rel.}}), \tag{1.3}$$

where ι is contraction. The one form $\iota_J \Phi(u_{[\phi]_{rel.}})$ is not holomorphic, but admits a residue nonetheless. By Theorem 1 there is a unique choice $[\overline{\phi}]_{rel.}$ such that the corresponding solution $\overline{u}$ is an absolute minimum of energy in the free homotopy class $[\overline{\phi}]$, and we use (1.3) to prove that $\overline{u}$ is the unique element in $\mathcal{H}arm_{\mathfrak{c},G}$ for which $\Phi(\overline{u})$ extends smoothly to all of Σ (§6). In other words, $\overline{u}$ is the unique critical point of $E : \mathcal{H}arm_{\mathfrak{c},G} \longrightarrow \mathbb{R}$. We refer to $\overline{u}$ as an *absolute minimizer* to distinguish it from the other *relative minimizers* in $\mathcal{H}arm_{\mathfrak{c},G}$.

As a corollary to (1.3), we prove the following formula for the Hessian of energy at $\overline{u}$. Given a path u_t through $u_0 = \overline{u}$ with derivative $J = \frac{d}{dt}\big|_{t=0} u_t$, we can define the Hopf differential of J by $\Phi(J) = \frac{d}{dt}\big|_{t=0} \Phi(u_t)$. We have

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(u_t) = \Re 2\pi i \operatorname{Res}|_{\overline{q}} \iota_J \Phi(J) \tag{1.4}$$

We use (1.4) to prove the non-degeneracy of the linearized residue map near $\overline{u}$ (§6), and use this to show that in the cone angle less than π case one can perturb in the direction of absolute minimizers as the geometric data varies.

In the closedness portion (section 7), we adapt standard methods for proving regularity of minimizing maps to the conic setting. First, we prove that the sup norm of the energy density of a minimizing map is controlled by the geometric data (see Proposition 7.4). This uses a standard application of the theorems of DiGiorgi-Nash-Moser and an extension of a Harnack inequality from [He]. It is noteworthy that if a conic metric is chosen on the domain, with cone point at the inverse image of the cone point of a the target via a minimizer from either of the main theorems, then the energy density is bounded from both above and below if and only if the cone angles are equal, and we work with such conformal metrics on the domain in what follows.

Control of the energy density near the (inverse image of) the cone points is insufficient, and to obtain stronger estimates we proceed by contradiction, employing a rescaling argument, and the elliptic regularity of b-differential operators, to produce a minimizing map of the standard round cone. We also classify such maps, and the map produced by rescaling is not among them; thus the desired bounds hold near the cone points (see Proposition 7.8).

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2. Setup and an example

In this section, we give the precise statements of the theorems and outline the method of proof.

Let Σ be a closed surface, $\mathfrak{p} = \{p_1, \dots, p_k\} \subset \Sigma$ a collection of k distinct points, and set

$$\Sigma_{\mathfrak{p}} := \Sigma - \mathfrak{p}. \tag{2.1}$$

Given a smooth metric G on $\Sigma_{\mathfrak{p}}$, let $u : \Sigma \longrightarrow \Sigma$ be a continuous map so that $u^{-1}(p)$ is a single point for all $p \in \mathfrak{p}$. Let

$$\mathfrak{p}' = u^{-1}(\mathfrak{p})$$

and assume that

$$u : \Sigma_{\mathfrak{p}'} \longrightarrow \Sigma_{\mathfrak{p}} \tag{2.2}$$

Is smooth. Let g be a smooth metric on $\Sigma_{\mathfrak{p}'}$. The differential du is a section of $\mathrm{T}^*\Sigma_{\mathfrak{p}'} \otimes u^*\mathrm{T}\Sigma_{\mathfrak{p}}$, which we endow with the metric $g \otimes u^*G$. We define the Dirichlet energy by

$$E(u, g, G) = \frac{1}{2}\int_{\Sigma_{\mathfrak{p}'}} \|du\|^2_{g\otimes u^*G}\, dVol_g = \frac{1}{2}\int_{\Sigma_{\mathfrak{p}'}} g^{ij} G_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j} dVol_g. \tag{2.3}$$

As we will see in (3.3) below, the value of the energy functional at such a triple depends only on the conformal class of g.

Assume for the moment that G is smooth on all of Σ. In this case critical points of this functional are smooth [Si] and satisfy that the *tension field*

$$\tau(u, g, G) := \operatorname{Tr} \nabla_{(u,g,G)}(du) = \Delta_g u^\gamma + {}^G\Gamma^\gamma_{\alpha\beta}\langle du^\alpha, du^\beta\rangle_g \tag{2.4}$$

is identically zero, where $\nabla_{(u,g,G)}$ is the Levi-Cevita connection on $\mathrm{T}^*\Sigma \otimes u^*\mathrm{T}\Sigma$ with metric $g^{-1} \otimes u^*G$, so $\operatorname{Tr}\nabla du \in \Gamma(u^*\mathrm{T}\Sigma)$. Here $\Gamma(B)$ denotes the space of smooth sections of a vector bundle $B \longrightarrow \Sigma$. In particular, $\tau(u, g, G)$ is a vector field over u, and it is minus the gradient of the energy functional in the following sense [EL].

Lemma 2.1 (First Variation of Energy)**.** *Let (M, g) and $(N, \widetilde{g})$ be smooth Riemannian manifolds, possibly with boundary, and let $u : (M, g) \longrightarrow (N, \widetilde{g})$ be a C^2 map. If u_t is a variation of C^2 maps through $u_0 = u$ and $\frac{d}{dt}\big|_{t=0} u_t = \psi$, then*

$$\left.\frac{d}{dt}\right|_{t=0} E(u_t, g, \widetilde{g}) = -\int_M \langle \tau(u, g, \widetilde{g}), \psi\rangle_{u^*\widetilde{g}} dVol_g + \int_{\partial M} \langle u_*\partial_\nu, \psi\rangle_{u^*\widetilde{g}} ds, \tag{2.5}$$

where ∂_ν is the outward pointing normal to ∂M and ds is the area form.

Returning to the case of our surface Σ, if G is instead only smooth on $\Sigma_{\mathfrak{p}}$, the map u in (2.2) still has a tension field

$$\tau(u, g, G) \in u^*\mathrm{T}\Sigma_{\mathfrak{p}}. \tag{2.6}$$

We will study the energy minimizing problem by finding zeros of τ.

2.1. **Example: Dirichlet problem for standard cones.** The standard cone of cone angle $2\pi\alpha$ is simply the sector in $\mathbb{R}^2$ of angle $2\pi\alpha$ with its boundary rays identified. Thus in polar coordinates $(\widetilde{r}, \widetilde{\theta})$ we can write this as a quotient $\mathbb{R}^+ \times [0, 2\pi\alpha]/\left((\widetilde{r}, 0) \sim (\widetilde{r}, 2\pi\alpha)\right)$, with the $g_\alpha := d\widetilde{r}^2 + \widetilde{r}^2 d\widetilde{\theta}^2$. Let $\widetilde{z} = \widetilde{r}e^{i\widetilde{\theta}}$, and set

$$\widetilde{z} = \frac{1}{\alpha} z^\alpha \tag{2.7}$$

Then, in terms of z,

$$g_\alpha = |z|^{2(\alpha-1)} |dz|^2, \tag{2.8}$$

and this expression is valid for all $z \in \mathbb{C}^*$. We denote this space by

$$\begin{aligned} C_\alpha &:= (\mathbb{C}, g_\alpha) \\ C_\alpha^* &:= C_\alpha - \{0\} \end{aligned} \tag{2.9}$$

Also let $D \subset \mathbb{C}$ be the standard disc of radius one, and set $D^* := D - \{0\}$. We will discuss the Dirichlet problem for harmonic maps from D to C_α. In this case, see (3.5), the tension field operator for a smooth map $u : D^* \longrightarrow C_\alpha^*$ is

$$\tau(u) = u_{z\overline{z}} + \frac{\alpha - 1}{u} u_z u_{\overline{z}}.$$

Therefore the identity map

$$\begin{aligned} id : D^* &\longrightarrow C_\alpha \\ z &\longmapsto z \end{aligned}$$

is harmonic on D^*. Given a map $\phi : \partial D \longrightarrow C_\alpha$ near to $id|_{\partial D}$, we would like to find a harmonic map

$$\begin{aligned} &u : D \longrightarrow C_\alpha \\ &\tau(u) = 0 \text{ on } D - u^{-1}(0) \\ &u|_{\partial D} = \phi \end{aligned} \tag{2.10}$$

Let $z = re^{i\theta}$. Initially, we consider τ acting on maps of the form

$$\begin{aligned} u(z) &= z + v(z) \\ v &\in r^{1+\epsilon} C_b^{2,\gamma}(C_\alpha(1)), \end{aligned} \tag{2.11}$$

where, given $v : D \longrightarrow \mathbb{C}$,

$$v \in r^c C_b^{2,\gamma}(D) \iff \begin{array}{c} r^{-c} v \text{ has uniformly bounded } C^{2,\gamma} \text{ norm on balls} \\ \text{of uniform size with respect to the rescaled} \\ \text{metric } g_\alpha / r^{2\alpha} = \frac{dr^2}{r^2} + d\theta^2. \end{array} \tag{2.12}$$

We will describe this space more precisely in section 2.3, but we mention now that v as in (2.11) satisfies $|v(z)| = \mathcal{O}(r^{1+\epsilon})$. We refer to the space of u as in (2.11) by $\mathcal{B}^{1+\epsilon}$. Let $\mathcal{B}_0^{1+\epsilon} \subset \mathcal{B}^{1+\epsilon}$ consist of u with $u|_{\partial D} = id|_{\partial D}$, i.e. those u whose v in (2.11) satisfy $v|_{\partial D}(z) \equiv 0$. Near id we can write $\mathcal{B}^{1+\epsilon}$ as a product of $\mathcal{B}_0^{1+\epsilon}$ and

$$\mathcal{E} := \left\{ \phi : \partial D \longrightarrow C_\alpha : \|\phi - id\|_{2,\gamma} < \epsilon \right\}$$

This can be done for example by picking a smooth radial cutoff function χ which is 1 near ∂D and defining $\mathcal{E} \longrightarrow \mathcal{B}^{1+\epsilon}$ be $\phi \longmapsto \phi(\theta)\chi(r)$; the addition map $\mathcal{E} \times \mathcal{B}_0^{1+\epsilon} \longrightarrow \mathcal{B}^{1+\epsilon}$ is a local diffeomorphism of Banach manifolds near (id, id).

The strategy now is to study τ acting on the space $\mathcal{B}^{1+\epsilon} \sim \mathcal{B}_0^{1+\epsilon} \times \mathcal{E}$. If the derivative of τ in the $\mathcal{B}_0^{1+\epsilon}$ direction were non-degenerate at id, then the zero set of τ near id would be a smooth graph over $\mathcal{E}$, *but this turns out to be false.* To fix this, we augment the domain of τ as follows. Define the spaces of conformal dilations and rotations

$$\begin{aligned} \mathcal{D} &= \{M_\lambda(z) = \lambda z : \lambda \in \mathbb{C}\} \\ \mathcal{T} &= \{T_w(z) = z - w : w \in \mathbb{C}\} \end{aligned}$$

and define two 2−dimensional spaces

$$\begin{aligned}\mathcal{D}_0 &= \left\{ \begin{array}{c} \text{any family } \widetilde{M}_\lambda \text{ parametrized by } \lambda \text{ near } 1 \in \mathbb{C} \\ \widetilde{M}_\lambda = M_\lambda \text{ near } 0 \text{ and vanishes on } \partial D \end{array} \right\} \\ \mathcal{T}_0 &= \left\{ \begin{array}{c} \text{any family } \widetilde{T}_w \text{ parametrized by } w \text{ near } 0 \in \mathbb{C} \\ \widetilde{T}_w = T_w \text{ near } 0 \text{ and vanishes on } \partial D \end{array} \right\}\end{aligned} \tag{2.13}$$

The dichotomy between cone angles less than π and between π and 2π now enters. We will show in §6 that for $\epsilon > 0$ sufficiently small

$$D_{id}\tau : T\left(\mathcal{B}_0^{1+\epsilon} \circ \mathcal{D}_0\right) \longrightarrow r^{-1+\epsilon} C_b^{0,\gamma}(D) \text{ is an isomorphism if } 0 < 2\pi\alpha < \pi \tag{2.14}$$

while

$$D_{id}\tau : T\left(\mathcal{B}_0^{1+\epsilon} \circ \mathcal{D}_0 \circ \mathcal{T}_0\right) \longrightarrow r^{-1+\epsilon} C_b^{0,\gamma}(D) \text{ is an isomorphism if } 2\pi > 2\pi\alpha > \pi \tag{2.15}$$

These two facts have obvious (and distinct) implications about solving (2.10). In the former case, for boundary values ϕ near id in $C^{2,\gamma}$ we can find solutions of the form $u(z) = \lambda z + v(z)$ while in the latter we must 'move the cone point,' so the solutions are in the form $u(z) = \lambda(z-w) + v(z-w)$.

As we will see in section 3.1, the precomposition of a harmonic map u with a conformal map is harmonic. Define $C_\alpha(1) = C_\alpha \cap \{z : |z| \leq 1\}$. Clearly $C_\alpha(1)$ is conformally equivalent to D, so for the moment we think of id not as a map from D to C_α but from $C_\alpha(1)$ to C_α, and applying the inverse of (2.7) to both spaces, think of id as the identity map on the wedge $\{r \leq 1, 0 \leq \theta \leq 2\pi\alpha\}$. (For simplicity, we have dropped the tildes from the notation.) We can think of a boundary map, ϕ near id as a map from the arc $\{r = 1, 0 \leq \theta \leq 2\pi\alpha\}$ into $\mathbb{C}$, satisfying the condition that

$$\phi(e^{2\pi\alpha i}) = e^{2\pi\alpha i}\phi(1). \tag{2.16}$$

We decompose ϕ in terms of the eigenfunctions of ∂_θ^2 which satisfy (2.16), i.e. we write $\phi(e^{i\theta}) = \sum_{j\in\mathbb{Z}} a_j e^{i(1+\frac{j}{\alpha})\theta}$. Assuming convergence, these are the values on the arc of a the sector of the harmonic map

$$u(z) = \sum_{j\geq 0} a_j z^{1+\frac{j}{\alpha}} + \sum_{j<0} a_j \overline{z}^{-1+\frac{j}{\alpha}}. \tag{2.17}$$

(In the flat metric $dr^2 + r^2 d\theta^2$ on the sector, the tension field of u is simply Δu, so by the decomposition $\Delta = 4\partial_z\partial_{\overline{z}}$, the sum of a conformal and an anti-conformal function is harmonic.)

Consider the coefficient a_{-1}. If $\alpha > 1/2$, the term $a_{-1}\overline{z}^{-1+\frac{1}{\alpha}}$ dominates $a_0 z$ as $z \to 0$, and it is easy to check that a power series as in (2.17) sufficiently close to id on the boundary passes to a map of the cone if and only if $a_{-1} = 0$. It is exactly the deformations $\mathcal{T}$ that eliminate the a_{-1} and give actual mappings of the cone. When $\alpha < 1/2$, any power series as in (2.17) gives a map on the cone, so for any sufficiently regular boundary data near id, there is a harmonic map of D^* with that boundary data.

To go deeper, as we show in section 5, the map u is minimizing among all maps with its boundary values if and only if the residue of the Hopf differential of u is zero, and a simple

computation shows that (in the case under consideration,)

$$\operatorname{Res}\Phi(u) = \overline{a_{-1}}\left(-1 + \frac{1}{\alpha}\right).$$

That is, *if $\alpha < 1/2$, a_{-1} is the obstruction to having a minimizer (w.r.t. the boundary values) on all of D, not just D^*.* Thus it is natural, in the case $\alpha < 1/2$, to ask if we can solve the augmented Dirichlet problem

$$\begin{aligned} u : D &\longrightarrow C_\alpha \\ \tau(u) &= 0 \text{ on } D - u^{-1}(0) \\ a_{-1} &= 0 \\ u|_{\partial D} &= \phi \end{aligned}. \tag{2.18}$$

In fact we can. By (2.14) there is a graph of solutions to (2.10) over $\mathcal{T}_0 \times \mathcal{E}$. The solutions lying over $\mathcal{T}_0 \times \{id\}$ solve (2.10) with identity boundary value. Call this space $\mathcal{S}$. In section **??**, we show that the map from $\mathcal{S}$ to a_{-1} has non-degenerate differential. Since it is a map of two dimensional vector spaces, it is an isomorphism, and we get a smooth graph of solutions to (2.18) over $\mathcal{T}_0 \times \mathcal{E}$, see Figure 1.

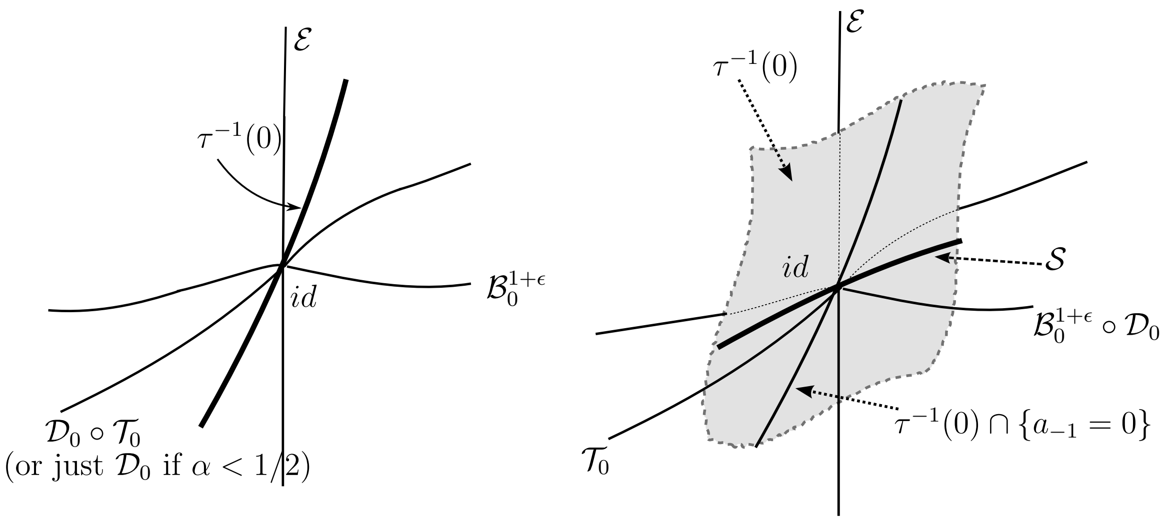

FIGURE 1. Solutions to the augmented equation (2.18) are found by applying the inverse function theorem to the map from $\mathcal{S}$ to a_{-1}.

2.2. **Conic metrics.** The simplest example of a conic metric is that of the standard cone C_α in (2.8). This motivates the following definition. Given $\alpha_j \in \mathbb{R}_+$ and $\nu_j > 0$, a Riemannian metric G on $\Sigma_{\mathfrak{p}}$ is said to have a conic singularity at $p_j \in \mathfrak{p}$ with cone angle $2\pi\alpha_j$ and type ν_j if there are conformal coordinates z centered at p_j such that

$$\begin{aligned} G &= c e^{2\lambda_j} |z|^{2(\alpha_j - 1)} |dz|^2 \\ \lambda_j = \lambda_j(z) &\in r^{\nu_j} C_b^{k,\gamma}(D(\sigma)) \\ c &> 0. \end{aligned} \tag{2.19}$$

Here $D(\sigma) = \{z : |z| < \sigma\}$. Obviously, in any other centered conformal coordinates $w = w(z)$, the metric will take the same form. For convenience, we single out those coordinates

for which $c = 1$, and we refer to these as *normalized* conformal coordinates. Given $\mathfrak{a} = (\alpha_1, \ldots, \alpha_k) \subset (0,1)^k$ and $\nu = (\nu_1, \ldots, \nu_k) \in \mathbb{R}_+^k$, we define

$$\mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a}) = \left\{ \begin{array}{c} C^{k,\gamma}_{loc} \text{ metrics on } \Sigma_{\mathfrak{p}} \text{ with cone points } \mathfrak{p} \\ \text{and cone angles } 2\pi\mathfrak{a} \text{ such that in (2.19)} \\ \text{we have } \lambda_j \in r^{\nu_j} C^{k,\gamma}_b(D(\sigma)) \text{ for some } \sigma. \end{array} \right\} \tag{2.20}$$

2.3. **The form of the minimizers.** We will look for harmonic maps which have specified behavior near the inverse image of the cone points. By the uniqueness theorem in section 5, all minimizers have this behavior, but we begin by stating it as an assumption.

Form 2.2. *We say that $u : (\Sigma, g) \longrightarrow (\Sigma, G)$ is in Form 2.2 (with respect to g and G) if*

1. *u is a homeomorphism of Σ homotopic to the identity.*
2. *If $\mathfrak{p}' = u^{-1}(\mathfrak{p})$, $u : \Sigma_{\mathfrak{p}'} \longrightarrow \Sigma_{\mathfrak{p}}$ is a $C^{2,\gamma}_{loc}$ diffeomorphism (see 2.1).*
3. *For each $p \in \mathfrak{p}$, if z is a conformal coordinate around $u^{-1}(p)$ w.r.t. g and by abuse of notation u is a conformal coordinate around p w.r.t. G, then*
$$u(z) = \lambda z + v(z)$$
where $\lambda \in \mathbb{C}^*$ and $v \in r^{1+\epsilon} C^{2,\gamma}_b(D(R))$ for some sufficiently small $\epsilon > 0$.

In words, in normal coordinates near p and $u^{-1}(p)$, u is a dilation composed with a rotation to leading order.

Remark 2.3. *When working with harmonic diffeomorphisms of surfaces it is often convenient to pull back the target metric, i.e. given $G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$ and*

$$u : (\Sigma_{\mathfrak{p}'}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G) \tag{2.21}$$

harmonic, to consider instead

$$id : (\Sigma_{\mathfrak{p}'}, g) \longrightarrow (\Sigma_{\mathfrak{p}'}, u^*G). \tag{2.22}$$

We will often make this assumption below. It should be noted that it is exactly Form 2.2 which guarantees that the pullback metric u^*G is a member of $\mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}', \mathfrak{a})$.

The space $C^{k,\gamma}_b(D(\sigma))$ was defined in the above as the space of functions with $C^{k,\gamma}$ norm uniformly bounded on balls of uniform size with respect to the rescaled metric $g/(e^{2\mu}|z|^{2\alpha}) = \frac{dr^2 + r^2 d\theta^2}{r^2}$, but here we give an alternative characterization that is easier to work with. Given $f : D(R) \longrightarrow \mathbb{C}$, let

$$\|f\|_{C^{0,\gamma}_b(D(R))} = \sup_{0<|z|\le R} |f| + \sup_{0<|z|,|z'|\le R} \frac{|f(z) - f(z')|}{|\theta - \theta'|^\gamma + \frac{|r-r'|^\gamma}{|r+r'|^\gamma}}. \tag{2.23}$$

(Here, as always, $z = re^{i\theta}, z' = r'e^{i\theta'}$.) Then

$$f \in C^{k,\gamma}_b(D(R)) \iff \left\|(r\partial_r)^i \partial_\theta^j f\right\|_{0,\gamma,D(R)} < \infty \text{ for all } i + j \le k. \tag{2.24}$$

Finally, define the weighted Hölder spaces by

$$\begin{aligned} f \in r^c C^{k,\gamma}_b(D(R)) &\iff r^{-c} f \in C^{k,\gamma}_b(D(R)) \\ \|f\|_{r^c C^{k,\gamma}_b(D(R))} = \left\|r^{-c} f\right\|_{C^{k,\gamma}_b(D(R))} &:= \sum_{i+j\le k} \left\|r^{-c}(r\partial_r)^i \partial_\theta^j f\right\|_{0,\gamma,D(R)} \end{aligned} \tag{2.25}$$

2.4. **Scalar curvature.** Let κ_G denote the gauss curvature of $G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$, and near $p \in \mathfrak{p}$, write $G = \rho |dz|^2$ with $\rho = e^{2\lambda} |z|^{2(\alpha-1)}$. There is a simple sufficient condition on ν which guarantees that $|\kappa_G| \le c < \infty$; one simply computes

$$\kappa_G = -\frac{1}{\rho}\partial_z \partial_{\overline{z}} \log \rho^2 = \frac{-4}{e^{2\lambda} |z|^{2\alpha}} (z\partial_z)(\overline{z}\partial_{\overline{z}})\mu. \tag{2.26}$$

Using $z\partial_z = \frac{1}{2}(r\partial_r - i\partial_\theta)$ we see that if $\mu \in r^{2\alpha} C_b^{2,\gamma}$, i.e. if G is type $\nu = 2\alpha$, then κ_G is a bounded function, and we will always make this assumption.

It will be important to have a somewhat stronger assumption about the metrics. We need the following

Definition 2.4. *A function $f : D(R) \longrightarrow \mathbb{C}$ is* ***polyhomogeneous*** *if f admits an asymptotic expansion*

$$f(r, \theta) \sim \sum_{(s,p) \in \widetilde{\Lambda}} r^s \log^p r a_{s,p}(\theta), \tag{2.27}$$

where $\widetilde{\Lambda} \subset \mathbb{R} \times \mathbb{N}$ is a discrete set for which each subset $\{s \le c\} \cap \widetilde{\Lambda}$ is finite. In this setup, $\widetilde{\Lambda}$ is called the 'index set' of f, and the symbol $\sim$ means that

$$f(r, \theta) - \sum_{(s,p) \in \widetilde{\Lambda}, N > s} r^s \log^p r a_{s,p}(\theta) = o(r^N)$$

Let

$$\mathcal{M}^{phg}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a}) = \left\{ h \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a}) \;\middle|\; \begin{array}{c} \text{near } p \in \mathfrak{p}, h = e^{2\mu} |z|^{2(\alpha-1)} |dz|^2 \\ \text{and } \mu \text{ is polyhomogeneous.} \end{array} \right\} \tag{2.28}$$

2.5. **Restatement of theorems.** Given two maps $u_i : \Sigma \longrightarrow \Sigma$, $i = 1, 2$, and a finite subset $\mathfrak{q} \subset \Sigma$, we define the equivalence relation

$$u_1 \sim_{\mathfrak{q}} u_2 \iff \begin{array}{c} u_1 \text{ is homotopic to } u_2 \text{ via} \\ F : [0,1] \times \Sigma \longrightarrow \Sigma \\ F_t(q) = u_1(q) \text{ for all } (t, q) \in [0,1] \times \mathfrak{q} \end{array}. \tag{2.29}$$

If $u_1 \sim_{\mathfrak{q}} u_2$, we say that the two maps are homotopic relative to $\mathfrak{q}$ (rel. $\mathfrak{q}$). For fixed u_1, the set of all u_2 satisfying (2.29) is referred to as the rel. $\mathfrak{q}$ homotopy class of u_1, and is denoted by $[u_1; \mathfrak{q}]$. By $u_1 \sim u_2$ we mean that u_1 and u_2 are homotopic with no restrictions on the homotopy.

Define

$$\begin{aligned} \mathfrak{p}_{<\pi} &= \{p_i \in \mathfrak{p} | 2\pi\alpha_i < \pi\} \\ \mathfrak{p}_{>\pi} &= \{p_i \in \mathfrak{p} | 2\pi\alpha_i > \pi\} \\ \mathfrak{p}_{=\pi} &= \{p_i \in \mathfrak{p} | 2\pi\alpha_i = \pi\} \end{aligned} \tag{2.30}$$

and assume that

$$\mathfrak{p}_{=\pi} = \varnothing.$$

We discuss the case $\mathfrak{p}_{=\pi} \neq \varnothing$ in section 8.

Theorem 3. *Assume that genus $\Sigma > 0$. For Σ and $\mathfrak{p}$ as above, let $G \in \mathcal{M}^{phg}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$ have cone angles less than 2π, and $\kappa_G \leq 0$. Let $\mathfrak{c}$ be a conformal class on Σ. Fix a (possibly empty) subset $\mathfrak{q} \subset \mathfrak{p}_{<\pi}$, let $w_0 : \Sigma \longrightarrow \Sigma$ be a homeomorphism.*

If $\mathfrak{q} \neq \varnothing$, set $\mathfrak{q}' := w_0^{-1}(\mathfrak{q})$. Then there exists a unique energy minimizing map u in $[w_0; \mathfrak{q}']$. Furthermore,

$$u^{-1}(p_i) \text{ is a single point for all } i = 1, \dots, k, \tag{2.31}$$

and

$$u : \Sigma - u^{-1}(\mathfrak{p}) \longrightarrow \Sigma - \mathfrak{p} \tag{2.32}$$

is a diffeomorphism.

If $\mathfrak{q} = \varnothing$, there exists an energy minimizing map u in $[w_0]$, unique up to precomposition with a conformal automorphism of $(\Sigma, \mathfrak{c})$. These also satisfy (2.31) and (2.32).

If $\Sigma = S^2$ then, assumptions as above, there is a unique energy minimizing map in the rel. $\mathfrak{q}$ homotopy class of w_0 provided $|\mathfrak{q}| \geq 3$. Again the map satisfies (2.31)—(2.32).

A simple argument using the isometry invariance of the energy functional allows us to reduce Theorem 3 to the case in which $\mathfrak{q}' = \mathfrak{q}$ and $w_0 \sim_{\mathfrak{q}} id$. In fact, if we prove Theorem 3 in this case, then given an arbitrary $w_0 \in \mathrm{Diff}(\Sigma)$ and a metric G with the properties in Theorem 3, let $\widetilde{u} : (\Sigma, (w_0^{-1})^* \mathfrak{c}) \longrightarrow (\Sigma, G)$ be the unique minimizer in $[id; \mathfrak{q}]$. Then since composition with an isometry leaves the energy unchanged, $u := \widetilde{u} \circ w_0^{-1} : (\Sigma, \mathfrak{c}) \longrightarrow (\Sigma, G)$ is the unique minimizer in $[w_0; \mathfrak{q}']$. The same argument with $\mathfrak{q} = \varnothing$ shows that we can reduce to the case $w_0 \sim_{\mathfrak{q}} id$. Thus we will always assume that $\mathfrak{q}' = \mathfrak{q}$.

Our approach to proving the theorem is to find homeomorphisms whose tension fields vanish away from $u^{-1}(\mathfrak{p})$. Among diffeomorphisms in Form 2.2 with vanishing tension field, minimizing energy in the sense of Theorem 3 turns out to be equivalent to a condition on the Hopf differential of u, which we define now. Given $u : \Sigma_{\mathfrak{p}'} \longrightarrow \Sigma_{\mathfrak{p}}$ with $\tau(u, g, G) = 0$ on $u^{-1}(\mathfrak{p})$, let $u^* G^\circ$ denote the g-trace-free part of $u^* G$. Among C^2 maps, away from the cone points

$$\tau(u, g, G) = 0 \implies \delta_g \left(u^* G^\circ \right) = 0, \tag{2.33}$$

where δ_g is the divergence operator for the metric g acting on symmetric $(0, 2)-$ tensors. Trace-free, divergence-free tensors are equal to the real parts of holomorphic quadratic differentials w.r.t. the conformal class $[g]$, so in conformal coordinates, we can write

$$u^* G = \lambda \left| dz \right|^2 + 2\Re \left(\phi(z) dz^2 \right) \tag{2.34}$$

where ϕ **is holomorphic**. Parting slightly with standard notation, e.g. from [W3], we use the symbol Φ to refer to the tensor which in conformal coordinates is expressed $\phi(z)dz^2$; this is called the **Hopf differential** of u. In section 5.1 we will show that, if $\mathfrak{p}'_{<\pi} = u^{-1}(\mathfrak{p}_{<\pi})$, then

$$\Phi(u) \text{ is holomorphic on } \Sigma - \mathfrak{p}'_{<\pi} \text{ with at most simple poles on } \mathfrak{p}'_{<\pi}.$$

Our proof of Theorem 3 then relies on the following lemma, proven in section 5.

Lemma 2.5. *Say genus $\Sigma > 0$. Given a harmonic diffeomorphism $u : (\Sigma_{\mathfrak{p}'}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G)$, for any $\mathfrak{q} \subset \mathfrak{p}_{<\pi}$, if $\mathfrak{q}' = u^{-1}(\mathfrak{q})$, then u is energy minimizing in its rel. $\mathfrak{q}'$ homotopy class if and only if the Hopf differential $\Phi(u)$ extends smoothly to all of $\Sigma_{\mathfrak{q}'}$.*

If $\Sigma = S^2$, the same is true so long as $|\mathfrak{q}| \geq 3$.

Remark 2.6. *If $\mathfrak{q} = \varnothing$, this lemma means that u is minimizing in its free homotopy class if and only if $\Phi(u)$ is smooth on all of Σ. We say that such maps are **absolute** minimizers.*

Thus a diffeomorphism with vanishing tension field on $\Sigma_{\mathfrak{p}'}$ is minimizing in the sense of Theorem 3 if and only if the residues of its Hopf differential vanish on $\mathfrak{p}'_{<\pi} - \mathfrak{q}$, i.e. when it solves the augmented equation

$$\begin{array}{rcll} u : \Sigma & \longrightarrow & \Sigma & \text{a homeomorphism} \\ u & \sim_{\mathfrak{q}} & id & \\ \tau(u, g, G) & = & 0 & \text{on } \Sigma_{\mathfrak{p}'} \\ \mathrm{Res}|_p \Phi(u) & = & 0 & \text{for each } p \in \mathfrak{p}'_{<\pi} - \mathfrak{q}. \end{array} \tag{HME($\mathfrak{q}$)}$$

Remark 2.7. *The residues of a holomorphic quadratic differential are only defined up to multiplication by an element of $\mathbb{C}^*$. Whether they are zero or not is well-defined, but since we will work with the residues directly, we will have to fix first conformal coordinates near each cone point.*

3. The harmonic map operator

In this section we discuss the global analysis of the map τ. We begin by discussing the invariance under conformal change of the domain metric.

3.1. **Conformal invariance.** Let $g, G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$, and suppose that u is a C^2 map of $\Sigma_{\mathfrak{p}}$. Suppose we have conformal expressions $G = \rho\,|du|^2$ at some $x \in \Sigma_{\mathfrak{p}}$ and $g = \sigma\,|dz|^2$ and near $u^{-1}(q)$. The energy density (the integrand in (2.3)) in conformal coordinates is

$$e(u, g, G)(z) := \frac{1}{2}\,\|du(z)\|^2_{g \otimes u^* G} = \frac{\rho(u(z))}{\sigma(z)}\left(|\partial_z u|^2 + |\partial_{\overline{z}} u|^2\right) \tag{3.1}$$

From this expression it is easy to verify that if

$$\begin{array}{c} (\Sigma_1, g) \xrightarrow{C} (\Sigma_1, g) \xrightarrow{u} (\Sigma_2, G) \\ z \longmapsto w = C(z) \longmapsto u(w) \end{array} \tag{3.2}$$

for arbitrary surfaces Σ_i, $i = 1, 2$ and C is conformal ($C^* g = e^{2\mu} g$), then

$$e(u \circ C, g, G)(z) = |\partial_z C(z)|^2\, e(u, C^* g, G)(z), \tag{3.3}$$

and so

$$E(u \circ C, g, G) = E(u, C^* g, G) = E(u, g, G). \tag{3.4}$$

A simple computation preformed e.g. in [SY] shows that

$$\boxed{\tau(u, g, G) = \frac{4}{\sigma}\left(u_{z\overline{z}} + \frac{\partial \log \rho}{\partial u} u_z u_{\overline{z}}\right)} \tag{3.5}$$

The tension field enjoys a point-wise conformal invariance; in the situation of (3.2), if one knows only at some particular point z_0 that $C_{\overline{z}}(z_0) = 0$, one can use this formula to chech that

$$\tau(u \circ C, g, G) = \tau(u, C^* g, G) \circ C \tag{3.6}$$

3.2. **Global analysis of τ.** Given a diffeomorphism $\widetilde{u}_0 : (\Sigma_{\mathfrak{p}}, g_0) \longrightarrow (\Sigma_{\mathfrak{p}'}, \widetilde{G})$ solving, pull back $\widetilde{G}$ as in Remark 2.3 to obtain that $id = u_0 : (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G)$ is a diffeomorphism solving (HME($\mathfrak{q}$)) in Form 2.2. To avoid tedious repetition below, we give a name to our main assumption on the metrics and maps.

Assumption 3.1. *(1) $G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$ with $\alpha_j < 1$ for each $2\pi\alpha_j \in \mathfrak{a}$, and $\nu_j > 2\alpha_j$ (see section 2.4)*
(2) g is also in $\mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$.

By (**??**), if we let τ act on a space of varying maps and metrics, then the range of this operator depends on the domain, and therefore τ is most naturally viewed as a section of the bundle. We will now define precisely the domain of τ and the vector bundle E in which it takes values.

Fix coordinates z_j near each $p_j \in \mathfrak{p}$, conformal with respect to g, and let $z_j = r_j e^{i\theta_j}$. Continuing to abuse notation, let u_j be conformal coordinates for G near p_j. (We will often omit j from the notation when it is understood that we work with a fixed cone point.) First, given any u in Form 2.2 we define $r^c \mathcal{X}_b^{k,\gamma}(u)$ for any $c \in \mathbb{R}^k$ by

$$\psi \in r^c \mathcal{X}_b^{k,\gamma}(u) \iff \begin{cases} \psi \in \Gamma(u^* T\Sigma_{\mathfrak{p}}) \\ \psi \in C^{k,\gamma}_{loc} \text{ away from } u^{-1}(\mathfrak{p}) \\ \psi \in r^{c_j} C_b^{k,\gamma}(D(\sigma)) \text{ near } p_j \in \mathfrak{p} \end{cases}, \tag{3.7}$$

for some $\sigma > 0$.

Remark 3.2. *We will often write $r^{1+\epsilon}\mathcal{X}_b^{k,\gamma}(u)$ for a positive number ϵ, by which we mean $r^c\mathcal{X}_b^{k,\gamma}(u)$ where $c_j = 1+\epsilon$ for all j. Given $\delta \in \mathbb{R}$, by $c > \delta$ we mean that $c_j > \delta$ for all j.*

- Let $\mathcal{B}^{1+\epsilon}(u_0)$ be the space of perturbations of u_0 defined by
$$\mathcal{B}^{1+\epsilon}(u_0) = \{\exp_{u_0} \psi \mid \psi \in r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}(u_0), \|\psi\|_{r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}} < \delta\}. \tag{3.8}$$
The $\delta > 0$ is picked small enough so that all the maps in this space are diffeomorphisms of $\Sigma_{\mathfrak{p}}$ and will be left out of the notation. Note that
$$\mathrm{T}_{u_0}\mathcal{B}^{1+\epsilon}(u_0) \simeq r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}(u_0) \tag{3.9}$$
- We also need a space of automorphisms of Σ, analogous to those in 2.13, that are locally conformal near $\mathfrak{p}$ with respect to g. Let $\mathcal{D}$ be a $2\,|\mathfrak{p}|$-dimensional space of diffeomorphisms of Σ parametrized by an open set $0 \in U \in \mathbb{C}^{|\mathfrak{p}|}$, so that $\lambda \in U$ corresponds to a map M_λ with
$$M_\lambda(z_j) = \lambda_j z_j \text{ near } p_j. \tag{3.10}$$
Similarly, let $\mathcal{T}$ be a $2\,|\mathfrak{p}'|$-dimensional space of diffeomorphisms of Σ parametrized by an open set $0 \in V \in \mathbb{C}^{|\mathfrak{p}'|}$, so that $w \in V$ corresponds to a map T_w with
$$T_w(z_j) = z_j - w_j \text{ near } p_j. \tag{3.11}$$
To be precise, we take these locally defined maps and extend them to diffeomorphisms of Σ in such a way that they are the identity outside a compact set, and in particular

such that $T_0 = id$. For any subset $\widetilde{\mathfrak{p}} \subset \mathfrak{p}$, define $\mathcal{T}_{\widetilde{\mathfrak{p}}} \subset$ by the condition that that $w_i = 0$ for $p_i \notin \widetilde{\mathfrak{p}}$. This can be done in such a way that

$$\mathcal{T} = \mathcal{T}_{>\pi} \circ \mathcal{T}_{<\pi} \circ \mathcal{T}_{=\pi}. \tag{3.12}$$

Finally, set

$$\mathcal{C} := \mathcal{D} \circ \mathcal{T} \tag{3.13}$$

- Finally, given $h_0 \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$, define a subspace

$$\mathcal{M}^*_{k,\gamma,\nu}(h_0, \mathfrak{p}, \mathfrak{a}) \subset \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$$

as follows. Let $D_j = \{z_j \leq 1\}$ where again z_j are conformal coordinates for h_0 near p_j.

$$\mathcal{M}^*_{k,\gamma,\nu}(h_0, \mathfrak{p}, \mathfrak{a}) = \left\{ h \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a}) \,\middle|\, \begin{array}{c} id|_{D_j} : (D_j, h_0) \longrightarrow (\Sigma, h) \\ \text{is conformal for all } j \end{array} \right\} \tag{3.14}$$

In words, this means the conformal coordinates for h_0 near $\mathfrak{p}$ **are** conformal coordinates for h_0. This definition may seem arbitrary, but as we will see in section 9, it is motivated by the requirement that τ be a continuous map.

To clarify the relationship between $\mathcal{M}^*_{k,\gamma,\nu}(h_0, \mathfrak{p}, \mathfrak{a})$ and $\mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$, we can construct, locally near h_0, a smooth injection from an open ball $\mathcal{U}$ in the latter space into $\mathrm{Diff}_0(\Sigma; \mathfrak{p})$ times the former by uniformizing locally around $\mathfrak{p}$. To be precise, given any $h \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$, let $v_h : D_j \longrightarrow (D_j, h)$ be the solution to the Riemann mapping problem normalized by the condition that $v_h(0) = 0$ and $v_h(1) = 1$. Let $\chi : D_j \longrightarrow \mathbb{R}$ be a smooth cutoff function with $\chi \equiv 1$ near 0 and $\chi \equiv 0$ near ∂D_j. Consider the map

$$\widetilde{v}_h(z) = \begin{cases} \chi(z_j) v_h(z_j) + (1 - \chi(z_j))\, z_j & \text{on } D_j \\ id & \text{elsewhere} \end{cases}$$

Then $\widetilde{v_h}$ is well-defined, and is a diffeomorphism if, for $\|v_h - id\|_{C^\infty(D)} < \epsilon$. We have

Lemma 3.3. *The map — defined locally near h_0 — given by*

$$\begin{aligned} \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a}) &\longrightarrow \mathrm{Diff}_0(\Sigma; \mathfrak{p}) \times \mathcal{M}^*_{k,\gamma,\nu}(h_0, \mathfrak{p}, \mathfrak{a}) \\ h &\longmapsto (\widetilde{v}_h, \widetilde{v}_h^* h) \end{aligned}$$

is an isomorphism onto its image on a ball near h_0.

Proof. This follows immediately from the fact that the solution to the Riemann mapping problem has C^∞ norm controlled by the distance from h to h_0 □

Note that, by construction $id : (\Sigma, h_0) \longrightarrow (\Sigma, \widetilde{v}_h^* h)$ is conformal near $\mathfrak{p}$.

With these definitions, we consider

$$\begin{aligned} \tau : \left(\mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0) \circ \mathcal{C}\right) \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, u^{-1}(\mathfrak{p}), \mathfrak{a}) \times \mathcal{M}^*_{2,\gamma,\nu}(G, \mathfrak{p}, \mathfrak{a}) &\longrightarrow \mathsf{E} \\ (u, g, G) &\longmapsto \tau(u, g, G), \end{aligned} \tag{3.15}$$

where $\pi : \mathsf{E} \longrightarrow \left(\mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0) \circ \mathcal{C}\right) \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, u^{-1}(\mathfrak{p}), \mathfrak{a}) \times \mathcal{M}^*_{2,\gamma,\nu}(G, \mathfrak{p}, \mathfrak{a})$ is the bundle whose fibers satisfy

$$\pi^{-1}(u \circ C, g, G) = r^{1+\epsilon-2\mathfrak{a}} \mathcal{X}^{0,\gamma}_b(u)$$

In section 9, we will prove

Proposition 3.4. *Let (u_0, g_0, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. Then the map (3.15) is C^1.*

3.3. **Proof of Theorem 3.** Let $\mathfrak{c}$ be any conformal structure on Σ and $G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$ any metric satisfying the hypotheses of the theorems. Given $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{q} \neq \varnothing$ (resp. $\mathfrak{q} = \varnothing$), we would like to find a map $u : (\Sigma, \mathfrak{c}) \longrightarrow (\Sigma, G)$ that minimizes energy in the rel. $\mathfrak{q}$ homotopy class of the identity (resp. the free homotopy class of the identity.) Let $\mathfrak{c}_0 := [G]$ be the conformal class of G, and let $\mathfrak{c}_t$, $t \in [0, 1]$ be a smooth path of conformal structures from $\mathfrak{c}_0$ to $\mathfrak{c}_1 = \mathfrak{c}$. (That the space of conformal structures is connected follows immediately from the convexity of the space of metrics.) Finally, define

$$\mathcal{H}(\mathfrak{q}) = \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \text{there is a map } u_t : (\Sigma, \mathfrak{c}_t) \longrightarrow (\Sigma, G) \\ \text{so that } (u_t, \mathfrak{c}_t, G) \text{ satisfies (HME}(\mathfrak{q})). \end{array} \right\}$$

In the remainder of this paper, we will prove

Proposition 3.5. *If genus $\Sigma > 0$, $\mathcal{H}(\mathfrak{q})$ is closed, open, and non-empty. If $\Sigma = S^2$, then the same is true provided $|\mathfrak{q}| \geq 3$.*

Remark 3.6. *The set is non-empty; it contains 0, since $\mathfrak{c}_0 = [G]$ and thus the identity is conformal. Degree one conformal maps are global energy minimizers in their homotopy classes,* [EL].

We can now prove that Theorem 3 is true, at least up to the proofs of the preceding proposition and Proposition 5.2, where uniqueness is shown.

Proof of Theorem 3. The content of Proposition 5.2 is that solutions to (HME($\mathfrak{q}$)) are minimizing in their rel. $\mathfrak{q}$ (or free if $\mathfrak{q} = \varnothing$) homotopy classes, and that such minimizers are unique in the appropriate sense. Thus it suffices to solve the equation, but Proposition 3.5 implies that a solution always exists. □

4. Openness via non-degeneracy

Our proof that $\mathcal{H}(\mathfrak{q})$ is open relies on two non-degeneracy results. We describe these now, and then use them to prove openness.

4.1. **Non-degeneracy of τ.** Let (u_0, g, G) solve (HME($\mathfrak{q}$)) with u_0 in Form 2.2. Let u_t be a C^1 path in $\mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0) \circ \mathcal{D} \circ \mathcal{T}_{>\pi}$ and write $\psi := \dot{u}_0$. Define

$$L\psi := \left.\frac{d}{dt}\right|_{t=0} \tau(u_t, g, G). \tag{4.1}$$

By (3.9) and the fact that $u_0 = id$, the domain of L is

$$\mathrm{T}\left(\mathcal{B}^{1+\epsilon} \circ \mathcal{D} \circ \mathcal{T}_{>\pi}\right) = r^{1+\epsilon}\mathcal{X}^{2,\gamma}_b + \mathrm{T}_{id}\mathcal{D} + \mathrm{T}_{id}\mathcal{T}_{>\pi}.$$

We will show that the linearization of τ is transverse to those conformal Killing fields of g which lie in its natural range. A conformal Killing field for g is a vector field C satisfying $\mathcal{L}_C g = \mu g$ for some function μ, where $\mathcal{L}$ denotes the Lie derivative. This is the derivative of the conformal map equation $F_t^* g = e^{\mu_t} g$ for some family F_t with $F_0 = id$. It is well known

that for surfaces the conformal killing fields are exactly the tangent space to the identity component of the conformal group,

$$\mathcal{C}onf_0 = \left\{C : (\Sigma, g) \longrightarrow (\Sigma, g) : C^* g = e^{2\mu} g\right\} \tag{4.2}$$

This space contains only the identity map if genus $\Sigma > 1$ and is two or three dimensional if the genus is 1 or 0, respectively. We have

Proposition 4.1. *Notation as above,*

$$L\left(T\left(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + T_{id}\mathcal{D} + T_{id}\mathcal{T}_{>\pi}\right)\right) \oplus \left(T\mathcal{C}onf_0 \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}\right) = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}. \tag{4.3}$$

In words, τ in (3.15) is transver to whatever conformal Killing fields lie in the range. 6.

4.2. **Non-degeneracy of the residue map.** To describe the second non-degeneracy result, we begin by describing the space of harmonic maps near a given solution to (HME($\mathfrak{q}$)) with fixed geometric data. Let (u_0, g, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. By the previous section, there is an open set $\mathcal{U} \subset \mathcal{T}_{<\pi} \times \mathcal{M}^*_{2,\gamma}(G, \mathfrak{p}, \mathfrak{a})$ and a map graphing zeros of the tension field operator,

$$\begin{aligned} \mathcal{S} : \mathcal{U} &\longrightarrow \mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0) \circ \mathcal{D} \circ \mathcal{T}_{>\pi} \\ (T, G) &\longmapsto u \text{ with } \tau(u, g, G) = 0, \end{aligned} \tag{4.4}$$

so that $u = \widetilde{u} \circ D \circ T' \circ T$, where $T' \in \mathcal{T}_{>\pi}, D \in \mathcal{D}, \widetilde{u} \in \mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0)$.

Given $\widetilde{\mathfrak{q}} \subset \mathfrak{p}_{<\pi}$, set

$$\mathcal{T}_{\widetilde{\mathfrak{q}}} := \left\{T_w : w_i = 0 \text{ for } p_i \notin \widetilde{\mathfrak{q}}\right\},$$

c.f. (3.12). We take $\widetilde{\mathfrak{q}} = \mathfrak{p}_{<\pi} - \mathfrak{q}$. Given the identification of $\mathcal{T}_{\mathfrak{p}_{<\pi}-\mathfrak{q}}$ with a ball $U \subset \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|}$ around the origin, if U is a sufficiently small we can define (locally near u_0 the $2\,|\mathfrak{p}_{<\pi} - \mathfrak{q}|$)-dimensional manifold of harmonic maps fixing $\mathfrak{q}$,

$$\mathcal{H}arm_{\mathfrak{q}} = \left\{\mathcal{S}(T_w, G) : w \in U\right\}, \tag{4.5}$$

Also, let

$$u_w := \mathcal{S}(T_w, G).$$

We denote the tangent space to $\mathcal{H}arm_{\mathfrak{q}}$ by

$$\boxed{\mathcal{J}_{\mathfrak{q}} := T_{u_0}\mathcal{H}arm_{\mathfrak{q}}.} \tag{4.6}$$

We identify this space with $\mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|}$ by setting

$$J_w := \left.\frac{d}{dt}\right|_{t=0} u_{tw}. \tag{4.7}$$

Clearly $LJ_w = 0$ on $\Sigma_{\mathfrak{p}}$. Consider the residue map,

$$\begin{aligned} \mathrm{Res}_{u_w^{-1}(\mathfrak{p}_{<\pi}-\mathfrak{q})} : \mathcal{H}arm_{\mathfrak{q}} &\longrightarrow \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|} \\ u_w &\longmapsto \mathrm{Res}|_{u_w^{-1}(\mathfrak{p}_{<\pi}-\mathfrak{q})}\,\Phi(u_w), \end{aligned}$$

where $\Phi(u_w)$ is the Hopf differential defined in (2.33)-(2.34). (The J_w also have Hopf differentials, defined by $\Phi(J_w) = \frac{d}{dt}\big|_{t=0}\Phi(u_{tw})$.) Differentiating Res at u_0 gives

$$\begin{aligned} D\,\mathrm{Res}_{\mathfrak{p}_{<\pi}-\mathfrak{q}} : \mathcal{J}_{\mathfrak{q}} &\longrightarrow \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|} \\ J_w &\longmapsto \mathrm{Res}(\Phi(J_w)). \end{aligned} \tag{4.8}$$

We will prove the following

Proposition 4.2. *If genus $\Sigma > 1$, the map (4.8) is an isomorphism. If genus $\Sigma = 1$, then the space $\mathcal{H}arm_{\mathfrak{q}}$ decomposes near id as*

$$\mathcal{H}arm_{\mathfrak{q}} = \mathcal{H}arm'_{\mathfrak{q}} \circ \mathcal{C}onf_0, \tag{4.9}$$

and

$$D\,\mathrm{Res}_{\mathfrak{p}_{<\pi}-\mathfrak{q}} : T\mathcal{H}arm'_{\mathfrak{q}} \longrightarrow \mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|} \text{ is injective.} \tag{4.10}$$

4.3. $\mathcal{H}(\mathfrak{q})$ **is open.** We will now use the two non-degeneracy results just discussed to prove openness.

Proposition 4.3. *Propositions 4.1 and 4.2 imply that $\mathcal{H}(\mathfrak{q})$ is open.*

Proof. Given $t_0 \in \mathcal{H}(\mathfrak{q})$, let $g_0 \in \mathfrak{c}_{t_0}$ be a metric satisfying Assumption 3.1. (See section 3.3 for definitions.) We now use Lemma 3.3; there is a small $\delta > 0$ such that for $t \in (t_0 - \delta, t_0 + \delta)$ there is a path of diffeomorphisms $\widetilde{v}_t$, all isotopic to the identity, such that the pullback conformal structures $\widetilde{v}_t^* \mathfrak{c}_t$ have the property that $id : (\Sigma, \mathfrak{c}_{t_0}) \longrightarrow (\Sigma, \widetilde{v}_t^* \mathfrak{c}_t)$ is conformal near $\mathfrak{p}$. Let $\widetilde{g}_t \in \widetilde{v}_t^* \mathfrak{c}_t$ be any family of metrics that equal g_{α_j} on the conformal ball D_j near p_j. The point is that this can be done uniformly for t near t_0 since all the conformal structures $\widetilde{v}_t^* \mathfrak{c}_t$ are equal there. Thus $\widetilde{g}_t \in \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$. We claim that there is a unique solution $\widetilde{u}_t : (\Sigma, \widetilde{g}_t) \longrightarrow (\Sigma, G)$ to (HME($\mathfrak{q}$)). Assuming this for the moment, the proof is finished, since $u_t := \widetilde{u}_t \circ \widetilde{v}_t^{-1} : (\Sigma, \left(\widetilde{v}_t^{-1}\right)^* \widetilde{g}_t) \longrightarrow (\Sigma, G)$ is a relative minimizer and $\left(\widetilde{v}_t^{-1}\right)^* \widetilde{g}_t \in \mathfrak{c}_t$.

The proposition is proven modulo the existence of $\widetilde{u}_t$, which we now prove using Propositions 4.1 and 4.2.

If genus $\Sigma > 1$, then Proposition 4.1 states that the differential of the map (3.15) at a solution (u_0, g_0, G) to (HME($\mathfrak{q}$)) in the direction of $\mathcal{B}^{1+\epsilon} \circ \mathcal{D} \circ \mathcal{T}_{>\pi}$ is an isomorphism. This together with the fact that τ is C^1 (Proposition 3.4) and the Implicit Function Theorem shows that the zero set of τ near (u_0, g_0, G) is a smooth graph over $\mathcal{T}_{<\pi} \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$. The map $\mathcal{S}$ in (4.4) realizes the manifold of relative minimizers as a graph over an open set $\mathcal{U} \subset \mathcal{T}_{<\pi} \times \mathcal{M}^*_{2,\gamma,\nu}(G, \mathfrak{p}, \mathfrak{a})$. We now use Proposition 4.2. The map $D(\mathrm{Res}_{\mathfrak{p}_{<\pi}-\mathfrak{q}} \circ \mathcal{S})$ is an isomorphism from $\mathrm{T}_{id}\mathcal{T}_{\mathfrak{p}_{<\pi}-\mathfrak{q}}$ to $\mathbb{C}^{|\mathfrak{p}_{<\pi}-\mathfrak{q}|}$. Thus the set $\mathcal{U} \cap \left\{\mathrm{Res}^{-1}_{\mathfrak{p}_{<\pi}-\mathfrak{q}}(0)\right\}$, which by Lemma 2.5 consists of rel. $\mathfrak{q}$ minimizers, is a graph over an open set $\mathcal{V} \subset \mathcal{T}_{\mathfrak{p}_{<\pi}-\mathfrak{q}} \times \mathcal{M}^*_{2,\gamma,\nu}(G, \mathfrak{p}, \mathfrak{a})$ sufficiently close to $\{id\} \times G$. This proves the existence of the $\widetilde{u}_t$.

Now suppose genus $\Sigma = 1$. Let $\{C_i\}$ be a basis for $\mathrm{T}\mathcal{C}onf_0 \cap r^{1+\epsilon-2\mathfrak{a}} \mathcal{X}_b^{0,\gamma}$. Note that

$$\mathrm{T}\mathcal{C}onf_0 \cap r^{1+\epsilon-2\mathfrak{a}} \mathcal{X}_b^{0,\gamma} = \begin{cases} \mathrm{T}\mathcal{C}onf_0 & \text{if} \quad \mathfrak{p}_{<\pi} = \varnothing \\ \{0\} & \text{if} \quad \mathfrak{p}_{<\pi} \neq \varnothing \end{cases} \tag{4.11}$$

This follows immediately from the fact that conformal Killing fields are nowhere vanishing.

Thus if $\mathfrak{p}_{<\pi} \neq \varnothing$ we again have surjective differential, and as in the genus > 1 case, rel. $\mathfrak{q}$ minimizers forms a graph over $\mathcal{T}_{\mathfrak{p}_{<\pi}-\mathfrak{q}} \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$. We lift to the universal cover of Σ, which we can take to be $\mathbb{C}$, and let z denote the coordinate there. By integrating around a fundamental domain, we claim that if $\phi_w dz^2 = \Phi(J_w)$ then

$$\sum_{p_i \in \mathfrak{p}_{<\pi}} \mathrm{Res}|_{p_i} \phi_w = 0. \tag{4.12}$$

This follows immediately from the fact that the deck transformations are $z \mapsto z + z_0$ for some z_0, so ϕ_w is actually a periodic function with respect to the deck group. Define a subset $V \subset \mathbb{C}^{|\mathfrak{p}_{<\pi} - \mathfrak{q}|}$ by $V = \operatorname{span}\langle(1, \dots, 1), (i, \dots, i)\rangle$. Thus by (4.12), $V^\perp = D\,\mathrm{Res}(T\mathcal{H}arm_{\mathfrak{q}})$, where the orthocomplement is taken with respect to the standard hermitian inner product on $\mathbb{C}^{|\mathfrak{p}_{<\pi} - \mathfrak{q}|}$. Now by Proposition 4.2 and the fact that $\mathrm{T}\mathcal{C}onf_0$ has one complex dimension,

$$D\,\mathrm{Res} : T\mathcal{H}arm'_{\mathfrak{q}} \longrightarrow V^\perp$$

is an isomorphism, and again the existence of the $\widetilde{u}_t$ follows from the Implicit Function Theorem.

We now tackle the case: $\mathfrak{p}_{<\pi} = \varnothing$ and genus $= 1$. In this case $\mathcal{T}_{>\pi} = \mathcal{T}$. Let $\mathcal{C}onf(g)$ denote the identity component of the conformal group of g. Consider the quotient bundle $\widetilde{\mathsf{E}}$ whose fiber over (u, g, G) is given by E_u / V_u, where

$$V_u := u_* \mathrm{T}\mathcal{C}onf(g) \subset r^{1-\epsilon-2\mathfrak{a}} \mathcal{X}_b^{2,\gamma}(u)$$

Let $\pi : \mathsf{E} \longrightarrow \widetilde{\mathsf{E}}$ be the projection. Proposition 4.1 immediately implies that the differential of the composition $\pi \circ \tau$ in the direction of $\mathcal{B}^{1+\epsilon} \circ \mathcal{D} \circ \mathcal{T}$ is an isomorphism. By the Implicit Function Theorem, the zero set of $\pi \circ \tau$ near (u_0, g, G) is a smooth graph over $\mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$, so for each $g \in \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$ there is a map $u = \widetilde{u} \circ D \circ T$ with $\mathcal{T}$ such that

$$\tau(u, g, G) \in V_u. \tag{4.13}$$

We will show that (4.13) implies that $\tau(u, g, G) = 0$. Suppose that $\tau(u, g, G) = u_* C$ for some $C \in \mathrm{T}\mathcal{C}onf(g)$. Let $f_t \subset \mathcal{C}onf_0$ be a family with $\frac{d}{dt}\big|_{t=0} f_t = C$. By the conformal invariance of energy, (3.4), we have.

$$\left.\frac{d}{dt}\right|_{t=0} E(u \circ f_t, g, G) = 0 \tag{4.14}$$

On the other hand we will show using (3.4) that

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} E(u \circ f_t, g, G) &= \int_\Sigma \langle \tau(u, g, G), u_* C\rangle \sqrt{g} dx \\ &= \|u_* C\|^2_{L^2}, \end{aligned} \tag{4.15}$$

with L^2 norm as in (6.28). Some care is needed in the proof since in general the boundary term in (2.1) can be singular. We postpone the rigorous computation to section 6.6, where several similar computations are done at once. Using the fact that the solutions are a graph over $\mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$ to conclude that the $\widetilde{u}_t$ exist as in the previous cases.

Finally, suppose $\Sigma = S^2$. Then again $\mathrm{T}\mathcal{C}onf_0 \cap r^{1+\epsilon-2\mathfrak{a}} \mathcal{X}_b^{0,\gamma} = \varnothing$ since the only conformal Killing field vanishing at three points is identically zero. The proof then proceeds as in the previous cases.

□

5. Uniqueness and convexity

The main result of this section is Proposition 5.2, which states that if (u, g, G) is a solution to (HME($\mathfrak{q}$)) in Form 2.2 satisfying Assumption 3.1, then, up to conformal automorphism, u is uniquely energy minimizing in its rel. $\mathfrak{q}$ homotopy class.

5.1. **The Hopf differential.** Let (u, g, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. In conformal coordinates, a trivial computation using (3.1) yields

$$u^*G = e(u)\sigma\,|dz|^2 + 2\Re\rho(u)u_z\overline{u}_z dz^2. \tag{5.1}$$

where $g = \sigma\,|dz|^2$. If we let u^*G° denote the trace free part of u^*G w.r.t. g, i.e. $u^*G^\circ = u^*G - \frac{1}{2}\left(\operatorname{tr}_g u^*G\right) g$, then $u^*G^\circ = 2\Re\Phi(u)$ where $\Phi(u)$ is the Hopf differential

$$\Phi(u) := \phi(z)dz^2 = \rho(u)u_z\overline{u}_z dz^2. \tag{5.2}$$

It follows directly (see section 9 of [S]) that for $z_0 \in \Sigma_{\mathfrak{p}}$

$$\begin{aligned} \tau(u, g, G)(z_0) = 0 &\implies \partial_{\overline{z}}\phi(z_0) = 0 \\ \partial_{\overline{z}}\phi(z_0) = 0 \quad \& \quad J(u)(z_0) \neq 0 &\implies \tau(u, g, G)(z_0) = 0, \end{aligned} \tag{5.3}$$

where $J(u)(z_0)$ is the Jacobian determinant of u. This means that, among local diffeomorphisms, the vanishing of the tension field is equivalent to the holomorphicity of the (locally defined) function ϕ. By Form 2.2, near $p \in \mathfrak{p}$ we have $u(z) = \lambda z + v(z)$ with $\lambda \in \mathbb{C}^*$ and $v(z) \in r^{1+\epsilon}C_b^{2,\gamma}$ for some $\epsilon > 0$. By the definition of $C_b^{2,\gamma}$ from the previous section, and the fact that $\partial_z = \frac{1}{2z}\left(r\partial_r - i\partial_\theta\right)$, we see that

$$\phi(z) = \left(|\lambda z|^{2(\alpha-1)} + o(|z|^{2(\alpha-1)})\right)(\lambda + o(1))\,\mathcal{O}(|z|^\epsilon) = \mathcal{O}(|z|^{-2+2\alpha+\epsilon}). \tag{5.4}$$

Since $-2+2\alpha+\epsilon > -2$, the function ϕ has at worst a simple pole at $z = 0$. If $\alpha \geq 1/2$ then $-2+2\alpha+\epsilon > -1$, so ϕ extends to a holomorphic function over p. Thus we have proven

Lemma 5.1. *Let $\Phi(u)$ be the Hopf differential of a solution (u, g, G) to (HME($\mathfrak{q}$)) in Form 2.2, with $G \in \mathcal{M}_{2,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$.*

$$\Phi(u) \text{ is holomorphic on } \Sigma - \mathfrak{p}_{<\pi} \text{ with at most simple poles on } \mathfrak{p}_{<\pi}.$$

5.2. **Uniqueness.** The main result of this section is the following.

Proposition 5.2. *Let (u, g, G) solve (HME($\mathfrak{q}$)) with u in Form 2.2, and assume that $\Phi(u)$ has non-trivial poles exactly at $\mathfrak{q} \subset \mathfrak{p}_{<\pi}$.*

Assume $\mathfrak{q} \neq \varnothing$. Then for any $w : \Sigma \to \Sigma$ with $w \sim_{\mathfrak{q}} u$ (see (2.29); in particular $w|_{\mathfrak{q}} = u|_{\mathfrak{q}}$) we have

$$E(u, g, G) \leq E(w, g, G)$$

with equality if and only if $u = w$.

If $\mathfrak{q} = \varnothing$ (i.e. (u, g, G) solves (HME($\mathfrak{q}$))) and $w : \Sigma \longrightarrow \Sigma$ satisfies the weaker condition $w \sim u$, then

$$E(u, g, G) \leq E(w, g, G)$$

with equality if and only if $u = w \circ C$ for $C \in \mathcal{C}onf_0$ (see (4.2)). In particular, if genus $\Sigma > 1$, equality holds if and only if $u = w$.

Proof. Assume $\mathfrak{q} = \varnothing$. We use a trick from [CH] to reduce to the smooth case.

First assume that the genus of $\Sigma > 1$. Let $\omega(z)\,|dz|^2$ be the unique constant curvature -1 metric in $[g]$. By section 5.1, for any $\epsilon > 0$ in local coordinates we can write

$$\begin{aligned} u^*G &= e(u)\sigma dz d\overline{z} + 2\Re\phi(z)dz^2 \\ &= \underbrace{\left(e(u)\sigma - \left(\epsilon\omega^2 + |\phi|^2\right)^{1/2}\right)|dz|^2}_{:=H_1} + \underbrace{\left(\epsilon\omega^2 + |\phi|^2\right)^{1/2}|dz|^2 + 2\Re\phi(z)dz^2}_{:=H_2} \end{aligned} \tag{5.5}$$

For ϵ sufficiently small H_1 is a metric on $\Sigma_{\mathfrak{p}}$; this follows from the fact that for u^*G to be positive definite we must have that $e(u)\sigma(z) > |\phi(z)|$. As for H_2, if $\mathfrak{q} = \varnothing$, ϕ extends smoothly $p \in \mathfrak{p}$, so H_2 is a smooth metric. It is slightly more involved but also trivial to verify that the Gauss curvature of H_2 satisfies

$$\kappa_{H_2} < 0. \tag{5.6}$$

See Appendix B of [CH] for the computation. Since the above characterization of $\tau(\widetilde{u}, g, \widetilde{G}) = 0$ in (5.3) is necessary and sufficient, we see that

$$\begin{aligned} id : (\Sigma, g) &\longrightarrow (\Sigma, H_1) \text{ is conformal} \\ id : (\Sigma, g) &\longrightarrow (\Sigma, H_2) \text{ is harmonic} \end{aligned}$$

From equation (2.3), for any $w : \Sigma \to \Sigma$ we have

$$E(w, g, G) = E(w, g, H_1) + E(w, g, H_2), \tag{5.7}$$

Unique minimization now follows from (5.7), the fact that degree one conformal maps are energy minimizing, and fact that a harmonic diffeomorphism into a negatively curved surface is uniquely energy minimizing in its homotopy class (see e.g. [Tr], [Har]).

If the genus of Σ is 1, then lifting to the universal cover $\mathbb{C}$, we obtain a harmonic map $\widetilde{u} : (\mathbb{C}, \pi^*g) \longrightarrow (\mathbb{C}, \widetilde{\pi}^*G)$, where π and $\widetilde{\pi}$ are conformal covering maps with respect to the standard conformal structure on $\mathbb{C}$. The metric $|dz|^2$ descends to $\Sigma_{\mathfrak{p}}$ and is in the unique ray of flat metrics in the conformal class of g. Here $\Phi(\widetilde{u}) = \widetilde{\phi}(\widetilde{z})d\widetilde{z}^2$ is defined globally on $\mathbb{C}$ and $\widetilde{\phi}$ is entire and periodic with respect to the deck transformations, hence bounded, hence constant. Write $\Phi(\widetilde{u}) = ad\widetilde{\widetilde{z}}^2$. For sufficiently small $\epsilon > 0$, we decompose

$$\widetilde{u}^*(\pi^*G) = \underbrace{\left(e(\widetilde{u})\sigma - \epsilon\right)|d\widetilde{z}|^2}_{:=\widetilde{K}_1} + \underbrace{\epsilon\,|d\widetilde{z}|^2 + 2\Re\left(ad\widetilde{z}^2\right)}_{:=\widetilde{K}_2} \tag{5.8}$$

Since $e(\widetilde{u})\sigma > 2\,|a|$ and $\left(e(\widetilde{u})\sigma\right)(\widetilde{z})$ is periodic, there is an ϵ so that the $\widetilde{K}_i$ are positive definite. If $K_i = \pi_*\widetilde{K}_i$, then $id : (\Sigma, g) \longrightarrow (\Sigma, K_i)$ is harmonic for both $i = 1, 2$. We now argue as above, invoking both the minimality of conformal maps, and the minimality up to conformal automorphisms for maps of flat surfaces. This completes the $\mathfrak{q} = \varnothing$ case.

If genus $= 0$, it is standard that $\Phi(u) \equiv 0$, and thus u is conformal away from $\mathfrak{q}$, hence globally conformal. Since conformal maps are energy minimizing this case is complete.

To relate the $\mathfrak{q} = \varnothing$ case to the $\mathfrak{q} \neq \varnothing$ case, we will use the following

Lemma 5.3. *Given a closed Riemann surface $R = (\Sigma, \mathfrak{c})$ (here $\mathfrak{c}$ is the conformal structure) with genus > 0 and any finite subset $\mathfrak{q} \subset \Sigma$, there is a finite sheeted conformal covering space $\pi : S \longrightarrow R$ which has non-trivial branch points exactly on $f^{-1}(\mathfrak{q})$. Furthermore, genus $S >$ genus Σ.*

If $R = S^2$, the above is true provided $|\mathfrak{q}| \geq 3$.

This is a well-known fact from the theory of Riemann surfaces. We include a sketch of a proof here for the convenience of the reader. First assume genus $\Sigma > 0$. For each $q \in \mathfrak{q}$, there is a branched cover $Y_q \longrightarrow R$ ramified above q, constructed as follows. The fundamental group of $R - \{q\}$ is a free group with $2g$ generators, where g is the genus. Let Z_q be any normal covering space of $R - q$ that is not a covering of R, i.e. let Z_q correspond to a normal subgroup of $\pi_1(R - \{q\})$ that is not the pullback of a normal subgroup in $\pi_1(R)$ via the induced map. Let Y_q be the unique closure of Z_q. Finally, let $\widetilde{S}$ be the composite of the Z_q, i.e. the covering of $R - \mathfrak{q}$ corresponding to the intersection of the groups corresponding to the Z_q. Then $\widetilde{S}$ is normal and has a unique closure S with a branched covering of R that factors through each of the $Y_q \longrightarrow R$. The deck transformations of $\widetilde{S}$ extend to S and by normality act transitively on the fibers, and therefore S is ramified exactly above $\mathfrak{q}$. [C]. If $\Sigma = S^2$ and $|\mathfrak{q}| \geq 3$, let $\mathfrak{q}_0 \subset \mathfrak{q}$ have exactly three points. There is a branched cover of Σ by a torus branched over $\mathfrak{q}_0$. Applying the previous argument to the torus and branching over all the lifts of cone points to the torus gives the result. [Ya]

Assuming $\mathfrak{q} \neq \varnothing$, let S a covering map branched exactly over $\mathfrak{q}$. (By our assumption that $|\mathfrak{q}| \geq 3$ in case $\Sigma = S^2$, such a cover exists.) and lift u to a map $\widetilde{u}$ so that

$$\begin{tikzcd}
(S, \pi^* g) \arrow[r, "\widetilde{u}"] \arrow[d, "\pi"] & (S, \pi^* G) \arrow[d, "\pi"] \\
(\Sigma_{\mathfrak{p}}, g) \arrow[r, "u"] & (\Sigma_{\mathfrak{p}}, G)
\end{tikzcd} \tag{5.9}$$

In particular, $\widetilde{u}^* \pi^* G = \pi^* u^* G$, so the Hopf differential of $\widetilde{u}$ is the pullback via the conformal map π of the Hopf differential of u. Pick any $p \in \mathfrak{q}$, and let $\phi(z)dz^2$ be a local expression of the Hopf differential of u in a conformal neighborhood centered at p. By assumption, ϕ has at most a simple pole at 0. Given $q \in \pi^{-1}(p)$, since q is a non-trivial branch point we can choose conformal coordinates $\widetilde{z}$ near q so that the map π is given by $\widetilde{z}^k = z$ for some $k \in \mathbb{N}$, $k > 1$. If

$$\phi(z)dz^2 = \left(\frac{a}{z} + h(z)\right) dz^2$$

where h is holomorphic, we have

$$\left(\frac{a}{z} + h(z)\right) dz^2 = k^2 \left(a\widetilde{z}^{k-2} + \widetilde{z}^{2k-2} h(\widetilde{z}^k)\right) d\widetilde{z}^2,$$

so $\Phi(\widetilde{u})$ is holomorphic on all of S, and therefore $\widetilde{u}$ solves (HME($\mathfrak{q}$)) with respect to the pullback metrics.

Note that by Lemma 5.3, the genus of S is at least 2, so we are in the right situation to apply the preceding argument. Specifically, suppose that $w \sim_{\mathfrak{q}} u$. Then the lifts $\widetilde{u}$ and $\widetilde{w}$ are homotopic and we have

$$\begin{aligned}
(\text{ \# of sheets }) E(u, g, G) &= E(\widetilde{u}, \pi^* g, \pi^* G) \\
&\leq E(\widetilde{w}, \pi^* g, \pi^* G) \\
&= (\text{ \# of sheets }) E(w, g, G)
\end{aligned}$$

with equality if and only if $\widetilde{u} = \widetilde{w}$, i.e. $u = w$.

$\square$

6. Linear Analysis

6.1. **The linearization.** We now explicitly compute the derivative of τ at u_0 in the $\mathcal{B}^{1+\epsilon}(u_0)$ direction. That is, given a solution (u_0, g_0, G) to (HME($\mathfrak{q}$)) with u_0 in Form 2.2, and a path $u_t \in \mathcal{B}^{1+\epsilon}(u_0)$ through u_0 with $\frac{d}{dt}\big|_{t=0} u_t = \psi$, we compute

$$L_{u_0,g_0,G}\psi := \frac{d}{dt}\bigg|_{t=0} \tau(u_t, g_0, G).$$

For any $u \in \mathcal{B}^{1+\epsilon}(u_0)$, near $p \in \mathfrak{p}$, write

$$\begin{aligned} u_0 &= \lambda z + v \\ u &= u_0 + \widetilde{v}, \\ v, \widetilde{v} &\in r^{1+\epsilon} C_b^{2,\gamma}. \end{aligned}$$

From (3.5), we have

$$\begin{aligned} \left(\frac{\sigma}{4}\right) \tau(u_0 + \widetilde{v}, g_0, G) &= \partial_z \partial_{\overline{z}} (u_0 + \widetilde{v}) + \frac{\partial \log \rho_0(u)}{\partial u} \partial_z (u_0 + \widetilde{v}) \partial_{\overline{z}} (u_0 + \widetilde{v}) \\ &= \tau(u_0, g_0, G) \\ &\qquad + \partial_z \partial_{\overline{z}} \widetilde{v} + \left(\frac{\partial \log \rho_0(u)}{\partial u} - \frac{\partial \log \rho_0(u_0)}{\partial u} \right) \partial_z u_0 \partial_{\overline{z}} u_0 \\ &\qquad + \frac{\partial \log \rho_0(u)}{\partial u} \left(\partial_z u_0 \partial_{\overline{z}} \widetilde{v} + \partial_z \widetilde{v} \partial_{\overline{z}} u_0 + \partial_z \widetilde{v} \partial_{\overline{z}} \widetilde{v} \right). \end{aligned}$$

Since $\tau(u_0, g_0, G) = 0$,

$$\begin{aligned} \left(\frac{\sigma}{4}\right) \tau(u_0 + \widetilde{v}, g, G) = \partial_z \partial_{\overline{z}} \widetilde{v} &+ \left(\frac{\partial \log \rho_0(u)}{\partial u} - \frac{\partial \log \rho_0(u_0)}{\partial u} \right) \partial_z u_0 \partial_{\overline{z}} u_0 \\ &+ \frac{\partial \log \rho_0(u)}{\partial u} \left(\partial_z u_0 \partial_{\overline{z}} \widetilde{v} + \partial_z \widetilde{v} \partial_{\overline{z}} u_0 + \partial_z \widetilde{v} \partial_{\overline{z}} \widetilde{v} \right). \end{aligned} \tag{6.1}$$

If we have u_t with $\frac{d}{dt}\big|_{t=0} u_t = \psi$, then

$$\frac{\sigma}{4} L\psi = \partial_z \partial_{\overline{z}} \psi + \frac{\partial \log \rho_0(u)}{\partial u} \left(\partial_z u_0 \partial_{\overline{z}} \psi + \partial_{\overline{z}} u_0 \partial_z \psi \right) + A\psi \tag{6.2}$$

where

$$\begin{aligned} \frac{\sigma}{4} A\psi &:= \partial_z u_0 \partial_{\overline{z}} u_0 \cdot \frac{d}{dt}\bigg|_{t=0} \frac{\partial \log \rho_0(u_t)}{\partial u} \\ &= 2 \partial_z u_0 \partial_{\overline{z}} u_0 \left(\frac{\partial^2 \mu}{\partial u \partial \overline{u}} \overline{\psi} + \frac{\partial^2 \mu}{\partial u^2} \psi \right) + \partial_z u_0 \partial_{\overline{z}} u_0 \frac{\alpha - 1}{u_0^2} \psi. \end{aligned} \tag{6.3}$$

As a preliminary to the analysis below, we can now see that L acts on weighted Hölder spaces (see (2.25)):

$$L : r^c \mathcal{X}_b^{2,\gamma} \longrightarrow r^{c-2\alpha} \mathcal{X}_b^{0,\gamma} \tag{6.4}$$

In fact, if we define the operator $\widetilde{L} := \frac{\sigma |z|^2}{4} L$ then

$$\widetilde{L}\psi = I(\widetilde{L})\psi + E(\psi) \tag{6.5}$$

where

$$I(\widetilde{L})\psi = (z\partial_z)\left(\overline{z}\partial_{\overline{z}}\right)\psi + (\alpha - 1)\,\overline{z}\partial_{\overline{z}}\psi \tag{6.6}$$

and E is defined (locally near $\mathfrak{p}$) by this equation. It follows immediately that

$$I(\widetilde{L}) : r^c C_b^{2,\gamma} \longrightarrow r^c C_b^{0,\gamma} \tag{6.7}$$

Furthermore, using $|z|\left|\frac{\partial\mu}{\partial u}\right| + |\partial_z v| + |\partial_{\overline{z}} v| = \mathcal{O}\left(|z|^\delta\right)$ we see that if $\psi \in r^c C_b^{2,\gamma}$ near $p \in \mathfrak{p}$ then

$$E(\psi) \in r^{c+\epsilon} C_b^{2,\gamma}. \tag{6.8}$$

From these two equations, (6.7) follows immediately.

For the arguments in section 7 we must also compute the mapping properties of the locally defined operator

$$Q\widetilde{v} = \tau(u_0 + \widetilde{v}, g_0, G) - L(\widetilde{v})$$

From (6.1)-(6.3), we have

$$\begin{aligned}
\frac{\sigma}{4}\left(\tau(u_0 + \widetilde{v}, g_0, G) - L(\widetilde{v})\right) &= \left(\frac{\partial \log \rho_0(u)}{\partial u} - \frac{\partial \log \rho_0(u_0)}{\partial u}\right)\partial_z u_0 \partial_{\overline{z}} u_0 \\
&\quad + \frac{\partial \log \rho_0(u)}{\partial u}\partial_z \widetilde{v}\partial_{\overline{z}}\widetilde{v} - A\widetilde{v} \\
&= \left(\frac{\partial \log \rho_0(u)}{\partial u} - \frac{\partial \log \rho_0(u_0)}{\partial u}\right)\partial_z u_0 \partial_{\overline{z}} u_0 \\
&\quad - \left.\frac{d}{dt}\right|_{t=0} \frac{\partial \log \rho_0(u_t)}{\partial u}\partial_z u_0 \partial_{\overline{z}} u_0 + \left(\frac{\partial\mu(u)}{\partial u} + \frac{\alpha - 1}{u}\right)\partial_z \widetilde{v}\partial_{\overline{z}}\widetilde{v}
\end{aligned}$$

From this formula and the fact that $\widetilde{v} \in r^{1+\epsilon} C_b^{2,\gamma}$, $\mu \in r^\epsilon C_b^{1,\gamma}$, and $\partial_{\overline{z}} u \in r^\epsilon C_b^{2,\gamma}$ we see that

$$\|Q(\widetilde{v})\|_{r^{1+2\epsilon-2\alpha} C_b^{2,\gamma}} < C\,\|\widetilde{v}\|_{r^{1+\epsilon} C_b^{1,\gamma}} \tag{6.9}$$

holds for $\epsilon > 0$ small, $k \in \mathbb{N}$ and $\gamma \in (0,1)$ arbitrary.

6.2. **The b-calculus package for L.** This section links the study of L to a large body of work on b-differential operators. For more detailed definitions and proofs of what follows we refer the reader to [Me]. Fixing conformal coordinates z_i near each $p_i \in \mathfrak{p}$, we make a smooth function $r : \Sigma \longrightarrow \mathbb{C}$ that is equal to $|z_i|$ in a neighborhood of each p_i. There is a smooth manifold with boundary $[\Sigma; \mathfrak{p}]$ and a smooth map $\beta : [\Sigma; \mathfrak{p}] \longrightarrow \Sigma$ which is a diffeomorphism from the interior of $[\Sigma; \mathfrak{p}]$ onto $\Sigma_{\mathfrak{p}}$, with $\beta^{-1}(\mathfrak{p}) = \partial[\Sigma; \mathfrak{p}] \simeq \cup_{i=1}^k S^1$, on which the map $r \circ \beta$ is smooth up to the boundary. The space $[\Sigma; \mathfrak{p}]$ is constructed by radial blow up. Finally, let

$$\mathcal{V}_b = \text{smooth vector fields on } [\Sigma; \mathfrak{p}] \text{ which are tangent to the boundary} \tag{6.10}$$

The Lie algebra $\mathcal{V}_b$ generates a filtered algebra of differential operators called b-differential operators.

For a more concrete definition, let B_i, $i = 1, 2$ be any two vector bundles over $[\Sigma; \mathfrak{p}]$. Then P is a differential b-operator of order N on sections of B if it admits a local expression

$$P = \sum_{i+j \leq N} a_{i,j}\,(r\partial_r)^i\,\partial_\theta^j \text{ where } a_I \in C^\infty\left([\Sigma; \mathfrak{p}]; End(B_1; B_2)\right) \tag{6.11}$$

near the boundary of $[\Sigma; \mathfrak{p}]$. We will call P b-**elliptic** if for all (r,θ) and $(\xi,\eta) \in \mathbb{R}^2 - \{(0,0)\}$,

$$\sum_{i+j\leq N} a_{i,j}(r,\theta)\xi^i\eta^j \text{ is invertible.}$$

Let (u, g, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. In (**??**), the operator $\widetilde{L}$ can be defined globally using our extension of r and the local conformal factor σ to positive global functions. We immediately see that $\widetilde{L}$ is an elliptic b-operator, and that E is a b-operator which, in local coordinates as in (6.11), has coefficients a_I tending to zero at a polynomial rate.

We will now describe the relevant properties of an elliptic b-differential operator in our context. To begin we define the set $\Lambda \subset \mathbb{C}$ of indicial roots of $\widetilde{L}$. Given $p \in \mathfrak{p}$, we define a set, Λ_p consisting of all $\zeta \in \mathbb{C}$ such that for some function $a = a(\theta) : S^1 \longrightarrow \mathbb{C}$ and some $p \in \mathbb{N}$

$$\widetilde{L} r^{\zeta} a(\theta) = o(r^{\zeta}). \tag{6.12}$$

The total set of indicial roots is

$$\Lambda = \{z \in \mathbb{C}^n : z_i \in \Lambda_{p_i} \text{ for some } i\}. \tag{6.13}$$

We will show in section 6.4 that Λ is a discrete subset of $\mathbb{R}$. For all $c \notin \Re\Lambda$ there is a parametrix $\widetilde{\mathcal{P}}_c$ for $\widetilde{L}$, i.e.

$$\begin{aligned} \widetilde{\mathcal{P}}_c \circ \widetilde{L} &= I - R_1 \\ \widetilde{L} \circ \widetilde{\mathcal{P}}_c &= I - R_2 \end{aligned} \tag{6.14}$$

for compact operators R_1 and R_2 on $r^c\mathcal{X}_b^{k,\gamma}$ and $r^c\mathcal{X}_b^{k-2,\gamma}$, respectively. This holds for all $k \geq 2$.

The spaces $r^c\mathcal{X}_b^{2,\gamma}$ can be replaced, in the sense that everything above is still true, by weighted b-Sobolev spaces, defined as follows. Let $d\mu$ be a smooth, nowhere-vanishing density on $\Sigma_{\mathfrak{p}}$ so that near $\mathfrak{p}$

$$d\mu = \frac{dr d\theta}{r}$$

Let $\mathcal{X}_{L^2_b(d\mu)}$ be the completion of $r^\infty\mathcal{X}_b^\infty$ (smooth vector fields vanishing to infinite order at $\mathfrak{p}$) with respect to the norm

$$\|\psi\|^2_{L^2_b(d\mu)} = \int_\Sigma \|\psi\|^2_{u^*G}\, d\mu$$

As in the Hölder case, $\psi \in r^c\mathcal{X}_{L^2_b(d\mu)}$ if and only if $r^{-c}\psi \in \mathcal{X}_{L^2_b(d\mu)}$, and $r^c\mathcal{X}_{L^2_b(d\mu)}$ is a Hilbert space with inner product

$$\langle \psi, \psi' \rangle_{r^c\mathcal{X}_{L^2_b(d\mu)}} = \langle r^{-c}\psi, r^{-c}\psi' \rangle_{\mathcal{X}_{L^2_b(d\mu)}}$$

We define the weighted b-Sobolev spaces by

$$\psi \in r^c\mathcal{X}_{H^k_b(d\mu)} \iff \begin{array}{c} \text{for all } k\text{-tuples } V_1, \ldots, V_k \in \mathcal{V}_b, \\ \nabla_{V_1} \ldots \nabla_{V_k}\psi \in r^c\mathcal{X}_{L^2_b(d\mu)} \end{array} \tag{6.15}$$

The operators in (6.14) also acting between weighted b-Sobolev spaces, and it follows by standard functional analysis that $\widetilde{L}$ is Fredholm, i.e. that it has closed range and finite

dimensional kernel and cokernel. This immediately implies that

$$L : r^c \mathcal{X}_{H_b^k(d\mu)} \longrightarrow r^{c-2\mathfrak{a}} \mathcal{X}_{H_b^{k-2}(d\mu)} \tag{6.16}$$

is Fredholm. Thinking of L as a map from the $r^c L_b^2(d\mu)$ orthocomplement of its kernel onto its range, we can define the generalized inverse

$$\begin{aligned} \mathcal{G}_c \circ L &= I - \pi_{ker} \\ L \circ \mathcal{G}_c &= I - \pi_{coker}, \end{aligned} \tag{6.17}$$

where π_{ker} (resp. π_{coker}) is the $r^c L_b^2(d\mu)$ (resp. $r^{c-2\mathfrak{a}} L_b^2(d\mu)$) orthogonal projections onto the kernel (resp. cokernel) of L. In particular

$$\begin{aligned} \mathcal{G}_c : r^{c-2\mathfrak{a}} \mathcal{X}_{H_b^{k-2}(d\mu)} &\longrightarrow r^c \mathcal{X}_{H_b^k(d\mu)} \\ \pi_{ker} : r^c \mathcal{X}_{H_b^k(d\mu)} &\longrightarrow r^c \mathcal{X}_{H_b^k(d\mu)} \\ \pi_{coker} : r^{c-2\mathfrak{a}} \mathcal{X}_{H_b^{k-2}(d\mu)} &\longrightarrow r^{c-2\mathfrak{a}} \mathcal{X}_{H_b^{k-2}(d\mu)} \end{aligned} \tag{6.18}$$

In fact this holds on the weighted b-Hölder spaces, so everywhere $r^{c'} \mathcal{X}_b^{k',\gamma}$ can replace $r^{c'} \mathcal{X}_{H_b^{k'}(d\mu)}$ (see [M], section 3 for more on the relationship between b-Hölder and b-Sobolev spaces). The fact that (6.18) holds for both b-Sobolev and b-Hölder spaces immediately implies the following lemma, which will be very useful below.

Lemma 6.1. *For any $c \notin \Lambda$,*

$$Ker\left(L : r^c \mathcal{X}_{H_b^2(d\mu)} \longrightarrow r^{c-2\mathfrak{a}} \mathcal{X}_{H_b^0(d\mu)}\right) = Ker\left(L : r^c \mathcal{X}_b^{2,\gamma} \longrightarrow r^{c-2\mathfrak{a}} X x_b^{0,\gamma}\right), \tag{6.19}$$

and $L(r^c \mathcal{X}_b^{2,\gamma}) \oplus W = r^{c-2\mathfrak{a}} \mathcal{X}_b^{0,\gamma}$, where

$$W = Coker\left(L : r^c \mathcal{X}_{H_b^2(d\mu)} \longrightarrow r^{c-2\mathfrak{a}} \mathcal{X}_{H_b^0(d\mu)}\right) := \left(L\left(r^c \mathcal{X}_{H_b^2(d\mu)}\right)\right)^{\perp} \tag{6.20}$$

where the orthocomplement is computed with respect to the $\langle , \rangle_{r^{c-2\mathfrak{a}} \mathcal{X}_{H_b^2(d\mu)}}$ inner product. This last equation says that both the image of (6.26) *and the image of* (6.16) *are complimented by $Im(\pi_{coker})$ in* (6.18)*.*

Finally, we will need to use the fact that approximate solutions admit partial expansions. To be precise, from Corollary 4.19 in [M], if $\psi \in r^c \mathcal{X}_b^{k,\gamma}$ and $L\psi \in r^{c+\delta-2\mathfrak{a}} \mathcal{X}_b^{k-2,\gamma}$ for $\delta > 0$, i.e. if $L\psi$ vanishes faster than the generically expected rate of $r^{c-2\mathfrak{a}}$, then ψ decomposes as

$$\psi = \psi_1 + \psi_2 \tag{6.21}$$

Where $\psi_1 \in r^{c+\delta} \mathcal{X}_b$ and ψ_2 admits an asymptotic expansion, meaning for some discreet $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}$ which intersects $\{\mathbb{R}z < C\}$ at a finite number of points,

$$\psi_2(z) = \sum_{(s,p) \in \mathcal{E}, c+\delta > s > c} a_{s,p}(\theta) r^s \log^p r, \tag{6.22}$$

where $a_{s,p} : S^1 \longrightarrow \mathbb{C}$. In particular, we have

Lemma 6.2. *Solutions to $L\psi = 0$ have complete asymptotic expansions, i.e. they are polyhomogeneous (see 2.4).*

This follows from the improvement of (6.18) in [M]; if $\mathcal{A}^{\mathcal{E}}_{phg}$ denotes the polyhomogeneous functions with index set $\mathcal{E}$, then, given $c' > c$,

$$\mathcal{G}_c : r^{c'-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} \longrightarrow r^c\mathcal{X}_b^{2,\gamma} + \left(\mathcal{A}^{\mathcal{E}}_{phg} \cap r^{c'}\mathcal{X}_b^{2,\gamma}\right) \tag{6.23}$$

We can also use (6.23) to prove polyhomogeneity for the solution u by applying the parametrix for the linearization of τ to the harmonic map equation. Recall from (6.9) that $\tau(u, g, G) = L(u) + Q(u)$. Since u is harmonic, we have $L(v) = -Q(v)$. We now want to let our parametrix $\mathcal{G}_c$, with $c_i = 1 + \epsilon$ for all i, act on both sides of this equation and to use the formula $\mathcal{G}_c L = I$. This yields

$$v = -\mathcal{G}Q(v) \tag{6.24}$$

We know that, locally $v \in r^{1+\epsilon}\mathcal{X}_b^{k,\gamma}$. From (6.9) we have

$$Q : r^{1+\epsilon}\mathcal{X}_b^{k,\gamma} \quad \longrightarrow \quad r^{1+2\epsilon-2\alpha}\mathcal{X}_b^{k-1,\gamma} \tag{6.25}$$

so by (6.23), $-\mathcal{G}Q(v) = v_1 + v_2$ where $v_1 \in \mathcal{X}^{1+2\epsilon}_{k+1,\gamma}$ and $v_2 \in \mathcal{A}^{\mathcal{E}}_{phg}$. The full expansion follows from induction; assuming that $v = v_{1,k} + v_{2,k}$, where $v_{1,k} \in \mathcal{X}^{1+k\epsilon}_{k+1,\gamma}$ and $v_{2,k} \in \mathcal{A}^{\mathcal{E}}_{phg}$ we apply the same reasoning to this decomposition yields to increase the vanishing order of the term $v_{1,k}$.

6.3. **Cokernel of $L : r^{1-\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$.** We continue to let (u_0, g_0, G) denote a solution to (HME($\mathfrak{q}$)) in Form 2.2 satisfying Assumption 3.1, and to let L denote the linearization of τ at u_0.

An important step in the proof of Proposition 4.1 is an accurate identification of 'the' cokernel of the map

$$L : r^{1-\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} \tag{6.26}$$

(note the '$-\epsilon$.') In this section we will prove the following lemma

Lemma 6.3.

$$r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} = L\left(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}\right) \oplus \mathcal{K}$$

where

$$\boxed{\mathcal{K} := \operatorname{Ker} L|_{r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{2,\gamma}},} \tag{6.27}$$

and this decomposition is L^2 orthogonal with respect to the inner product in (6.28).

Given two vector fields $\psi, \psi' \in \Gamma(u_0^* T\Sigma_{\mathfrak{p}})$ which are smooth and vanish to infinite order near $\mathfrak{p}$, we define the geometric L^2 pairing by

$$\langle \psi, \psi' \rangle_{L^2} := \int_\Sigma \langle \psi, \psi' \rangle_{u_0^*(G)}\, d\mu_g = \Re \int_M \psi\overline{\phi}\rho(u_0)\sigma(z)\,|dz|^2 \tag{6.28}$$

It is straightforward to check using $g, G \in \mathcal{M}_{k,\gamma,\nu}(\mathfrak{p}, \mathfrak{a})$ and $u_0 \sim \lambda z$ that if $\psi \in r^c\mathcal{X}_b^{0,\gamma}(u_0)$ and $\psi' \in r^{c'}\mathcal{X}_b^{0,\gamma}(u_0)$, then

$$c_i + c_i' > 2 - 4\alpha_i \forall i \implies \langle \psi, \psi' \rangle_{L^2} < \infty \tag{6.29}$$

Since they are necessary for subsequent functional analytic arguments, we define weighted L^2 spaces of vector fields:

Definition 6.4. *Let $\mathcal{X}_{L^2}$ denote the space with*

$$\psi \in \mathcal{X}_{L^2} \quad \Longleftrightarrow \quad \langle \psi, \psi \rangle_{L^2} = \|\psi\|^2_{L^2} < \infty$$

As in section 6.2, let r denote a positive function on $\Sigma_{\mathfrak{p}}$ with the property that, near $p_j \in \mathfrak{p}$, $r = r_j$ where $z_j = r_j e^{i\theta_j}$ for a centered conformal coordinate z_j. Given $a = (a_1, \dots, a_k)$, let r^a denote a smooth positive function equal to $r_i^{a_i}$ near p_i, and let $r^a \mathcal{X}_{L^2}$ be the space with

$$\psi \in r^a \mathcal{X}_{L^2} \quad \Longleftrightarrow \quad \langle r^{-a}\psi, r^{-a}\psi \rangle_{L^2} := \|\psi\|^2_{r^a L^2} < \infty$$

Let $r^a \mathcal{X}_{H^1_b}$ denote the space of sections ψ of $\Gamma(u_0^* T\Sigma)$ such that $\|\psi\|_{u_0^*(T\Sigma)}$ and also $\|\nabla\psi\|_{u_0^*(T\Sigma)}$ are in L^2_{loc}, and near any cone point $r\nabla_{\partial_r}\psi, \nabla_{\partial_\theta}\psi \in r^a \mathcal{X}_{L^2}$.

It is standard (see e.g. [EL]), that the linearization of τ in (4.1) is symmetric with respect to this inner product and appears in the formula of the Hessian of the energy functional near a harmonic map. For reference, we state this here as

Lemma 6.5 (Second Variation of Energy)**.** *Let (M, h) and $(N, \widetilde{h})$ be smooth Riemannian manifolds, possibly with boundary, and let $u_0 : (M, h) \longrightarrow (N, \widetilde{h})$ be a C^2 map satisfying $\tau(u_0, h, \widetilde{h}) = 0$. If u_t is a variation of C^2 maps through $u_0 = 0$ with $\frac{d}{dt}\big|_{t=0} u_t = \psi$, and L is the linearization of τ at u_0 (see (4.1)), then*

$$\boxed{L\psi = \nabla^*\nabla\psi + \operatorname{tr}_h R^{\widetilde{h}}(du, \psi)du},$$

where ∇ is the natural connection on $u_0^*(TN)$ induced by $\widetilde{h}$, and $R^{\widetilde{h}}$ is its curvature tensor. If ∂_ν denotes the outward pointing normal to ∂M, then

$$\begin{aligned} \frac{d^2}{dt^2} E(u_t)\bigg|_{t=0} &= \int_M \Big(\langle \nabla\psi, \nabla\psi \rangle_{u^*\widetilde{h}} + \operatorname{tr}_h R^{\widetilde{h}}(du, \psi, \psi, du) \Big) dVol_h \qquad (6.30) \\ &\quad + \int_{\partial M} \langle \nabla_{u_*\partial_\nu}\psi, \psi \rangle_{u^*\widetilde{h}} ds \\ &= -\langle L\psi, \psi \rangle_{L^2} + \int_{\partial M} \big(\langle \nabla_\psi \psi, u_*\partial_\nu \rangle_{u^*\widetilde{h}} + \langle \nabla_{u_*\partial_\nu}\psi, \psi \rangle_{u^*\widetilde{h}} \big) ds \qquad (6.31) \end{aligned}$$

Note that the boundary term in the last line can be expressed in terms of the Lie derivative

$$\int_{\partial M} \big(\langle \nabla_\psi \psi, u_*\partial_\nu \rangle_{u^*\widetilde{h}} + \langle \nabla_{u_*\partial_\nu}\psi, \psi \rangle_{u^*\widetilde{h}} \big) ds = \int_{\partial M} \mathcal{L}_{u_*\psi}\widetilde{h}(u_*\psi, u_*\partial_\nu) ds. \qquad (6.32)$$

We can also write down a necessary condition on the weights of variations in the b-Hölder spaces so that the Hessian of energy is given by the quadratic form corresponding to L. If $\psi \in r^{1-\mathfrak{a}+\epsilon}\mathcal{X}^{2,\gamma}_b$ and $u_t \in \mathcal{B}^{1-\mathfrak{a}+\epsilon}$ has $\frac{d}{dt}\big|_{t=0} u_t = \psi$, then

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(u_t, g, G) = -\langle L\psi, \psi \rangle_{L^2}$$

As for the symmetry of L with respect to the L^2 inner product, if $\psi \in r^c \mathcal{X}^{2,\gamma}_b$ and $\psi' \in r^{c'}\mathcal{X}^{2,\gamma}_b$ and $c_i + c_i' > 2\alpha_i - 2$, then

$$\langle L\psi, \psi' \rangle_{L^2} = \langle \psi, L\psi' \rangle_{L^2} \qquad (6.33)$$

The proofs are a simple excercise in counting orders of decay, using our assumptions on the u_0, g_0 and G (and $\Gamma \sim r^{-1}$.)

We will use the fact that L is symmetric with respect to the L^2 pairing to prove a relationship between the dimension of the cokernel and the dimension of the kernel of L on these weighted spaces. Given a constant $\delta \in \mathbb{R}$ and $c = (c_1, \dots, c_k) \in \mathbb{R}^k$, let $c + \delta = (c_1 + \delta, \dots, c_k + \delta)$.

Lemma 6.6. *Again, let (u_0, g_0, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. If $c \in \mathbb{R}^k$ and if $\psi \in \mathcal{X}^{c+\mathfrak{a}}_{H^1_b}$, $\psi' \in \mathcal{X}^{-c+\mathfrak{a}}_{H^1_b}$, we have*

$$\langle L\psi, \psi' \rangle = \langle \psi, L\psi' \rangle \tag{6.34}$$

and the bilinear form

$$\begin{aligned} r^{c+\mathfrak{a}}\mathcal{X}_{H^1_b} \times r^{-c+\mathfrak{a}}\mathcal{X}_{H^1_b} &\longrightarrow \mathbb{R} \\ \psi, \psi' &\mapsto \langle L\psi, \psi' \rangle \end{aligned}$$

is continuous and non-degenerate.

Proof. The fact that the equation $\langle L\psi, \psi' \rangle = \langle \psi, L\psi' \rangle$ holds for $\psi, \psi' \in r^\infty \mathcal{X}^{2,\gamma}_b$ follows from Lemma 6.5. All of our work up to this point then implies that both sides of the equation are continuous with respect to the stated norms, so part one is proven.

□

We can now prove Lemma 6.3.

Proof of Lemma 6.3. Lemma 6.6 immediately implies that

$$\operatorname{Ker} L|_{r^{c+\mathfrak{a}}\mathcal{X}_{H^1_b}} = \left(L \left(r^{-c+\mathfrak{a}} \mathcal{X}_{H^1_b} \right) \right)^\perp \tag{6.35}$$

where the right hand side is simply defined to be those $\psi \in r^{c+\mathfrak{a}}\mathcal{X}_{H^1_b}$ for which $\langle L\psi', \psi \rangle = 0$ for all $\psi' \in r^{-c+\mathfrak{a}}\mathcal{X}_{H^1_b}$. We need to relate this to the weighted b-Sobolev spaces in (6.15). Note that

$$r^c \mathcal{X}_{H^1_b(d\mu)} = r^{c+2\mathfrak{a}-1}\mathcal{X}_{H^1_b}.$$

It follows from this that

$$\begin{aligned} \operatorname{Ker} L|_{r^{c+\mathfrak{a}}\mathcal{X}_{H^1_b}} &= \operatorname{Ker} \left(L : r^{c-\mathfrak{a}+1}\mathcal{X}_{H^2_b(d\mu)} \longrightarrow r^{c-2\mathfrak{a}}\mathcal{X}_{H^0_b(d\mu)} \right) \\ \left(L \left(r^{-c+\mathfrak{a}} \mathcal{X}_{H^1_b} \right) \right)^\perp &= \operatorname{Coker} \left(L : r^{-c-\mathfrak{a}+1}\mathcal{X}_{H^2_b(d\mu)} \longrightarrow r^{c-2\mathfrak{a}}\mathcal{X}_{H^0_b(d\mu)} \right) \end{aligned} \tag{6.36}$$

By 6.35 the two left hand sides are equal, thus are right hand sides, so by Lemma 6.1 we have that $L(r^{1-\epsilon}\mathcal{X}^{2,\gamma}_b) + \mathcal{K} = r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}^{0,\gamma}_b$, where

$$\mathcal{K} = \operatorname{Ker} \left(L : r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_{H^2_b(d\mu)} \longrightarrow r^{1+\epsilon-4\mathfrak{a}}\mathcal{X}_{H^0_b(d\mu)} \right). \tag{6.37}$$

The proof will be finished if we can show that the sum is L^2 orthogonal. To do this, we use the fact from the last paragraph of section 6.2 that each $\psi \in \operatorname{Ker} L|_{r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}^{2,\gamma}_b}$ in fact satisfies $\psi \in r^{c+1-\mathfrak{a}+\delta}\mathcal{X}^{2,\gamma}_b$ for some small $\delta > 0$ (proof: such ψ admits a polyhomogeneous expansion in the indicial roots.) By (6.29), we have that $\langle L\psi, \psi' \rangle$ is finite and Lemma 6.33, $\langle L\psi, \psi' \rangle = \langle \psi, L\psi' \rangle = 0$. This completes the proof.

□

6.4. **The indicial roots of L.** We now compute the indicial roots of L, defined in (6.12). It is sufficient to find all r-homogeneous solutions to

$$I(\widetilde{L})\widetilde{\psi} = 0,$$

where $I(\widetilde{L})$ defined as in (6.6). Fixing $p \in \mathfrak{p}$, we separate variables; let $\widetilde{\psi} = r^s a(\theta)$ be a solution, and write $a(\theta) = \sum a_j e^{ij\theta}$. We have

$$I(\widetilde{L}) r^s e^{ij\theta} = e^{ij\theta} r^s \left(s^2 + 2(\alpha - 1)s - j^2 - 2(\alpha - 1)j\right),$$

which equals to zero if and only if $s \in \{j, 2 - 2\alpha - j\}$. Setting

$$\Lambda_p = \bigcup_{j \in \mathbb{Z}} \{j, 2 - 2\alpha - j\} \tag{6.38}$$

the indicial roots of $\widetilde{L}$ is the union of these sets, (6.13).

We can now say something quite precise about the leading asymptotics of solutions

Lemma 6.7 (Leading order asymptotics)**.** *Notation as above, near a cone point p of cone angle $2\pi\alpha > \pi$, we have $u = \lambda z + v$ where $v \in r^{3-2\alpha} C_b^{2,\gamma} \cap \mathcal{A}_{phg}^{\mathcal{E}}$.*

Proof. We know that u is polyhomogeneous by the end of the section 6.2. By (6.1), the leading order term of v, call it v_0, satisfies $I(\widetilde{L})v_0 = 0$. Since $v_0 \in r^{1+\epsilon} C_b^{2,\gamma}$, this implies that it in fact vanishes to the first order indicial root bigger than $1 + \epsilon$. By (6.38), this is at least as big as $3 - 2\alpha$. $\square$

6.5. **Properties of Jacobi fields.** Solutions to $L\psi = 0$ are called *Jacobi fields*. Using the previous section's analysis, we can now describe the behavior of Jacobi fields in $\mathcal{K}$ (see (6.27)) near the conic set.

Lemma 6.8 (Structure of Singular Jacobi Fields)**.** *Let $\psi \in \mathcal{K}$ (see (6.27)), let $p_i \in \mathfrak{p}$, and let z be conformal coordinates near p_i. If $2\pi\alpha_i < \pi$, then*

$$\psi(z) = \mu_i z + \mathcal{O}(|z|^{1+\delta}) \tag{6.39}$$

for some $\mu_i \in C$, $\delta > 0$.

If $2\pi\alpha_i > \pi$, and the solution $u_0(z) = z + v$, then

$$\psi(z) = w_i + \frac{\overline{a_i}}{1 - \alpha} |z|^{2(1-\alpha)} + \mathcal{O}\left(|z|^{2(1-\alpha)+\delta}\right) \tag{6.40}$$

for some $w_i, a_i \in \mathbb{C}$, $\delta > 0$.

The choice of coefficient of $|z|^{2(1-\alpha)}$ simplifies subsequent formulae.

Proof. This follows essentially from the analysis in sections 6.2 and 6.4. An arbitrary solution $L\psi = 0$ has an expansion near each cone point. Fix $p \in \mathfrak{p}$. By (6.6), the lowest order homogeneous term, which we can write as $f(r, \theta) = r^\delta a(\theta)$ for $a : S^1 \longrightarrow \mathbb{C}$, must solve $I(\widetilde{L})f = 0$, and so δ must be an indicial root, i.e. $\delta \in \Lambda(p)$. If $p \in \mathfrak{p}_{<\pi}$, then the smallest such indicial root is 1. The eigenvector is λz, so the lemma is proven in this case. If $p \in \mathfrak{p}_{>\pi}$, the smallest such indicial root is 0, and the eigenvector for this indicial root is a constant complex number w_i.

We could continue the proof by analyzing (6.2), but we prefer to kill two birds with one stone by analyzing the Hopf differential of ψ. Solutions to $L\psi = 0$ also have holomorphic quadratic differentials: their coefficients are the linearization of the coefficients for harmonic maps in (5.2). Precisely, suppose $\frac{d}{dt}\big|_{t=0} u_t = \psi$, and define

$$\begin{aligned}\Phi(\psi) &:= \frac{d}{dt}\bigg|_{t=0} \rho(u_t)\partial_z u_t \partial_z \overline{u}_t dz^2 \\ &= \left(\left(\frac{\partial \rho}{\partial u}\psi + \frac{\partial \rho}{\partial \overline{u}}\overline{\psi} \right) \partial_z u_0 \partial_z \overline{u}_0 + \rho(u_0) \left(\partial_z \psi \partial_z \overline{u}_0 + \partial_z \overline{\psi} \partial_z u_0 \right) \right) dz^2\end{aligned} \tag{6.41}$$

Using (6.2) one can easily check that $\Phi(\psi)$ is holomorphic on $\Sigma_{\mathfrak{p}}$. Near $p \in \mathfrak{p}_{<\pi}$, it is easy to show that $\Phi(\psi)$ extends holomorphically over p. Near $p \in \mathfrak{p}_{>\pi}$, since $\psi = w + \psi_0$ for $\psi_0 \in r^c C^{2,\gamma}_b$ for some $c > 0$ and $v \in r^{3-2\alpha} C^{2,\gamma}_b$ (Lemma 6.7), $\Phi(\psi)$ has at most a simple pole. Suppose that

$$\mathrm{Res}|_p \Phi(\psi) = a \tag{6.42}$$

We claim that the only term in (6.41) that contributes to this residue is $\rho(u_0)\partial_z\overline{\psi}\partial_z u_0$. To see this, note that

$$\frac{\partial \rho}{\partial u} = \mathcal{O}\left(|z|^{2\alpha - 3}\right),$$

so by Form 2.2 and the assumption $\alpha > 1/2$

$$\frac{\partial \rho}{\partial u}\psi \partial_z u_0 \partial_z u_0 = \mathcal{O}\left(|z|^{2\alpha-3+1+\epsilon}\right) = \mathcal{O}\left(|z|^{-1+\delta}\right)$$

for some $\delta > 0$, so this term does not contribute to the residue. Similarly, $\frac{\partial \rho}{\partial \overline{u}}\overline{\psi}\partial_z u_0 \partial_z \overline{u}_0$ does not contribute. Using 6.7, the other term is term is

$$\rho(u_0)\partial_z\psi\partial_z\overline{u}_0 = \mathcal{O}\left(r^{2\alpha-2}\right)\mathcal{O}\left(r^{\delta-1}\right)\mathcal{O}\left(r^{2-2\alpha}\right) = \mathcal{O}\left(r^{\delta-1}\right)$$

Finally, let $|z|^\delta b(\theta)$ be the lowest order homogeneous term in $\psi_0 = \psi - w$. Then the leading order part of $\rho(u_0)\partial_z\overline{\psi}\partial_z u_0$ is $|z|^{2(1-\alpha)} \partial_z \overline{|z|^\delta b(\theta)}$, which equals to $\overline{|z|^\delta b(\theta)}$. This implies that

$$|z|^\delta b(\theta) = \frac{\overline{a}}{1-\alpha} |z|^{2(\alpha - 1)},$$

and the proof is complete.

□

Remark 6.9. *Note that, as a consequence of the proof, if we (easily) choose conformal coordinates so that u_0 in Form 2.2 has $\lambda = 1$, then*

$$\boxed{\mathrm{Res}|_p \Phi(\psi) = a} \tag{6.43}$$

Writing $\mathfrak{p}_{>\pi} = \{q_1, \dots, q_n\}$, the map

$$\begin{aligned}\mathrm{Res} : \mathcal{K} &\longrightarrow \mathbb{C}^{|\mathfrak{p}_{>\pi}|} \\ \psi &\longmapsto \mathrm{Res}\,\Phi(\psi) = (\mathrm{Res}|_{q_1}\Phi(\psi), \dots, \mathrm{Res}|_{q_n}\Phi(\psi))\end{aligned}$$

is obviously linear. We define a basis of $\mathcal{K}$, $J_{a^1}, \dots, J_{a^{m_1}}, C_1, \dots, C_{m_2}$ with $a^j \in \mathbb{C}^n$, so that $C_1, \dots, C_{m_2}$ is a basis of $\ker(\mathrm{Res} : \mathcal{K} \longrightarrow \mathbb{C}^{|\mathfrak{p}_{>\pi}|})$, i.e.

$$\mathrm{Res}\, C_j = 0 \in \mathbb{C}^n \tag{6.44}$$

and

$$\operatorname{Res} J_{a^j} = a^j \in \mathbb{C}^n. \tag{6.45}$$

It follows that the a^j are linearly independent.

6.6. **The C_j are conformal Killing fields.**

6.6.1. *The Hessian of energy on $\mathcal{K}$.* We will prove that the C_j are conformal Killing fields. The most important step in the proof is that they are zeros of the Hessian of the energy functional, and we begin by proving this.

Lemma 6.10. *Let (u_0, g_0, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. Let $C \in \mathcal{K}$ have $\operatorname{Res} C = 0 \in \mathbb{C}^n$. By Lemma 6.8, we can find $u_t \in \mathcal{B}^c \circ \mathcal{D} \circ \mathcal{T}_{>\pi}$, $\epsilon > 0$ so that $\frac{d}{dt}\big|_{t=0} u_t = C$, where*

$$\begin{gathered} c_i > 1 \textit{ for } p_i \in \mathfrak{p}_{<\pi} \\ c_i > 2 - 2\alpha_i \textit{ for } p_i \in \mathfrak{p}_{>\pi}. \end{gathered} \tag{6.46}$$

Then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g_0, G) = 0 \tag{6.47}$$

The proof hinges on a nice cancellation of boundary terms related to the conformal invariance of energy. We begin by proving equation (4.15) which illustrates this phenomenon in a simpler setting.

Proof of (4.15): We are given a $u \in \mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{D} \circ \mathcal{T}_{\mathfrak{p}_{>\pi}}$, and we assume that $\mathfrak{p}_{<\pi} = \varnothing$ and genus $\Sigma = 1$. In particular, near each $q \in u^{-1}(\mathfrak{p})$ we can choose conformal coordinates so that $u \sim \lambda z$. Let $C \in \mathrm{T}\mathcal{C}on f_0$. Choose $f_t \in \mathcal{C}on f_0$ with $\frac{d}{dt}\big|_{t=0} f_t = C$ and consider

$$\left.\frac{d}{dt}\right|_{t=0} E(u \circ f_t, g, G),$$

which is zero by (4.14). By lifting to the universal cover we can assume that

$$f_t(z) = z - tw$$

for some fixed w. Let

$$D_w(r) = \{z : |z - w| < r\}, \tag{6.48}$$

be the conformal disc (not necessarily a geodesic ball), so $f_t(D_{tw}(r)) = D_0(r)$. In particular, for $\delta > 0$ sufficiently small, $f_t(\Sigma - D_{tw}(\delta)) = \Sigma - D_0(\delta)$. For the moment we drop the geometric data from the notation and let

$$E(w, A) = \int_A e(w, g, G)\sqrt{g}dx$$

for any $A \subset \Sigma$. By the conformal invariance of the energy functional, we have

$$\begin{gathered} E(u \circ f_t, \Sigma - D_{tw}(\delta)) = E(u, \Sigma - D_0(\delta)) \\ E(u \circ f_t, D_{tw}(\delta)) = E(u, D_0(\delta)) \end{gathered}$$

for all t. Thus the functions $E(u \circ f_t, \Sigma)$, $E(u \circ f_t, \Sigma - D_{tw}(\delta))$, and $E(u \circ f_t, D_{tw}(\delta))$ are all constant in t. In particular

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} E(u \circ f_t, \Sigma) \\ &= \frac{d}{dt}\Big|_{t=0} E(u \circ f_t, \Sigma - D_{tw}(\delta)) + \frac{d}{dt}\Big|_{t=0} E(u \circ f_t, D_{tw}(\delta)) \\ &= \frac{d}{dt}\Big|_{t=0} E(u \circ f_t, \Sigma - D_{tw}(\delta)) \end{aligned}$$

We can also evaluate this last expression using the first variation formula (2.1) and the chain rule to get the expression

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} E(u \circ f_t, \Sigma - D_{tw}(\delta)) = &- \int_{\Sigma - D_0(\delta)} \langle \tau(u, g, G), u_* C \rangle_{u^*G} dVol_g \\ &+ \int_{\partial D_0(\delta)} \langle u_* \partial_\nu, u_* C \rangle_{u^*G} ds \\ &+ \frac{d}{dt}\Big|_{t=0} \left(\int_{\Sigma - D_{tw}(\delta)} e(u, g, G) \sqrt{g} dx \right) \end{aligned} \tag{6.49}$$

Here ∂_ν is the outward pointing normal from $\Sigma - D_0(\delta)$. The last integral satisfies

$$\frac{d}{dt}\Big|_{t=0} \left(\int_{\Sigma - D_{tw}(\delta)} e(u, g, G) \sqrt{g} dx \right) = - \int_{\partial D_{tw}(\delta)} e(u, g, G) \langle \partial_\nu, u_* C \rangle_g ds \tag{6.50}$$

To compare this integral to (6.49), we use the decomposition (5.1), $u^*G = e(u)g + u^*G^\circ$.

$$\int_{\partial D_0(\delta)} \langle u_* \partial_\nu, u_* C \rangle_{u^*G} ds = \int_{\partial D_0(\delta)} e(u, g, G) \langle \partial_\nu, u_* C \rangle_g ds + \int_{\partial D_0(\delta)} \langle \partial_\nu, u_* C \rangle_{u^*G^\circ} ds$$

By (6.50), the first term on the right exactly cancels the last term in (6.49). For the second term, note that although u^*G° is not holomorphic, the bound from (5.4) still holds, so

$$\int_{\partial D_0(\delta)} \langle \partial_\nu, u_* C \rangle_{u^*G^\circ} ds = \int_0^{2\pi} \mathcal{O}\left(r^{2\alpha - 2 + \epsilon} \right) r d\theta,$$

which goes to zero since $\alpha > 1/2$. Looking back at (6.49), we have proven that

$$\left(\frac{d}{dt}\Big|_{t=0} E(u \circ f_t, \Sigma - D_{tw}(\delta)) \right) + \int_{\Sigma - D_0(\delta)} \langle \tau(u, g, G), u_* C \rangle_{u^*G} dVol_g \to 0$$
$$\text{as } \delta \to 0$$

This proves (4.15)

□

Remark 6.11. *Note that this last proof works if $\alpha > 1/2$ is replaced by $\alpha \geq 1/2$. This will be important when we deal with that case in section 8.*

A similar cancellation of boundary terms will lead to the proof of Lemma (6.10). Here we take two derivatives, so the relevant boundary terms look slightly different. To illustrate

this, let $g = \sigma |dz|^2$ be a conformal metric on $\mathbb{C}$ with finite area, and let $T_t = z - tw$, and $\frac{d}{dt}\big|_{t=0} T_t = C(= w)$. Notation as above

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(T_t, \mathbb{C} - D_{tw}(\delta)) = 0$$

As above, a direct computation of the second derivative using (6.5) and the chain rule will produce boundary terms which must cancel one another. If we let $e_t = e(T_t, |dz|^2, g)$, and let ∂_{ν_t} be the outward pointing normal to $\mathbb{C} - D_{tw}(\delta)$, then a simple computation using (6.31)–(6.32) and the product rule shows that

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} E(T_t, \mathbb{C} - D_{tw}(\delta)) &= \frac{d^2}{dt^2}\bigg|_{t=0} E(T_t, \mathbb{C} - D_{tw}(\delta)) \\ &= \int_{\partial D_0(\delta)} \mathcal{L}_{\dot{T}_0} g\left(\dot{T}_0, \partial_\nu\right) ds - 2\int_{\partial D_0(\delta)} \dot{e}_0 \langle \dot{T}_0, \partial_\nu\rangle_g ds \\ &\quad - \int_{\partial D_0(\delta)} e_0 \left.\frac{d}{dt}\right|_{t=0} \langle \dot{T}_{-t}, \partial_{\nu_{-t}}\rangle_{g(T_{-t})} ds.\end{aligned} \tag{6.51}$$

Thus the expression on the right *must be equal to zero.*

Proof of Lemma 6.10. Assume that $u_0 = id$. By (3.1) and conformal invariance, we may replace g by $g/e(u_0)$ and assume that

$$e(u_0) \equiv 1. \tag{6.52}$$

We arrange it so that

$$\begin{aligned}&u_t = \widetilde{u}_t \circ T_t \\ &\widetilde{u}_t \in r^{2(1-\alpha)} C_b^{2,\gamma} \text{ near } p \in \mathfrak{p} \\ &T_t \in \mathcal{T}_{>\pi},\end{aligned}$$

with c as in (6.46), so $T_t(z_i) = z_i - tw_i$ near $p_i \in \mathfrak{p}_{>\pi}$. Define

$$D_{i,t}(\delta) = D_{tw_i}(\delta) \text{ in conformal coordinates } z_i \text{ near } p_i\ ,$$

where $D_{tw_i}(\delta)$ is the conformal disc defined in (6.48). We can then write

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(u_t, \Sigma) = \underbrace{\frac{d^2}{dt^2}\bigg|_{t=0} E\left(u_t, \Sigma - \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_{i,t}(\delta)\right)}_{:=A(\delta)} + \underbrace{\frac{d^2}{dt^2}\bigg|_{t=0} E\left(u_t, \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_{i,t}(\delta)\right)}_{:=B(\delta)}.$$

For the term $B(\delta)$ we can use conformal invariance

$$E\left(u_t, \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_t(\delta)\right) = E\left(\widetilde{u}_t, \bigcup_{p_i \in \mathfrak{p}_{>\pi}} T_t\left(D_t(\delta)\right)\right) = E\left(\widetilde{u}_t, \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_0(\delta)\right).$$

The fact that $\widetilde{u}_t \in r^{2-2\alpha_i+\epsilon} C_b^{2,\gamma}$ near $p_i \in \mathfrak{p}_{>\pi}$ and the boundary term computation from section 6.6.1 show that $B(\delta) \to 0$ as $\delta \to 0$. Note that for fixed t and δ the integrals $E\left(u_t, \Sigma - \bigcup_{p_i \in \mathfrak{p}_{>\pi}} D_{i,t}(\delta)\right)$ are improper since we did not delete balls around the points

$p \in \mathfrak{p}_{<\pi}$, but again the boundary contributions from deleted discs here do not contribute to $A(\delta)$. Let

$$\Sigma(\delta) := \Sigma - \cup D_{i,t}(\delta). \tag{6.53}$$

We use the same reasoning as in (6.51) and the fact that T_{-1} parametrizes the boundary of Σ to deduce that

$$\begin{aligned} A(\delta) = \int_{\Sigma(\delta)} \mathcal{L}_C G(C, \partial_\nu) ds - 2 \int_{\Sigma(\delta)} \dot{e}_0 \langle \dot{T}_0, \partial_\nu \rangle_g ds \\ - \int_{\Sigma(\delta)} e_0 \left. \frac{d}{dt} \right|_{t=0} \langle \dot{u}_{-t}, \partial_{\nu_{-t}} \rangle_{g(T_{-t})} ds, \end{aligned}$$

where we have once again set $e_t = e(u_t)$. At this point we use that C is a Jacobi field, so in conformal coordinates $g = \sigma \, |dz^2|$,

$$\mathcal{L}_C G = \dot{e}_0 g + 2 \underbrace{\Re \phi(z) dz^2}_{\Phi(C)}.$$

We also have that

$$\operatorname{Res} C = 0 \implies \phi(z) dz^2 \text{ is smooth on all of } \Sigma.$$

Plugging in and separating everything from the Hopf differential bit, we have (using (6.52))

$$\begin{aligned} A(\delta) = \sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} 2 \left(\Re \phi(z) dz^2 \right) (C, \partial_r) \, \delta d\theta \\ + \sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} \left(\dot{e}_0 \langle C, \partial_r \rangle_g - 2 \dot{e}_0 \langle \dot{T}_0, \partial_r \rangle_g - \left. \frac{d}{dt} \right|_{t=0} \langle \dot{u}_{-t}, \partial_{r_{-t}} \rangle_{g(T_{-t})} \right) \delta d\theta \end{aligned} \tag{6.54}$$

Since ϕ is bounded, the first term on the right hand side goes to zero as $\delta \to 0$.

The point now is that up to terms vanishing with δ the bottom line consists exactly of the cancelling terms from above (6.51), proving Lemma 6.10. To show this, we start with the rightmost term. Since $\ddot{T} \equiv 0$ near $p \in \mathfrak{p}_{>\pi}$ and $\ddot{\tilde{u}}_0 \in r^{2-2\mathfrak{a}+\epsilon} \mathcal{X}_b^{2,\gamma}$, we see that

$$\begin{aligned} \int_{\partial\Sigma(\delta)} e_0 \left. \frac{d}{dt} \right|_{t=0} \langle \dot{u}_{-t}, \partial_{\nu_{-t}} \rangle_{g(T_{-t})} ds &= \sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} \left. \frac{d}{dt} \right|_{t=0} \left(\langle \dot{T}_t + \dot{\tilde{u}}_t, \partial_{r_{-t}} \rangle_{g(T_{-t})} \right) \delta d\theta \\ &= \sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} \left. \frac{d}{dt} \right|_{t=0} \langle \dot{T}_t, \partial_{r_{-t}} \rangle_{g(T_{-t})} \delta d\theta \\ &\quad + \sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} \left. \frac{d}{dt} \right|_{t=0} \left(\langle \dot{\tilde{u}}_t, \partial_{r_{-t}} \rangle_{euc} \sigma(T_{-t}) \right) \delta d\theta \\ &= \sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} \left. \frac{d}{dt} \right|_{t=0} \langle \dot{T}_t, \partial_{r_{-t}} \rangle_{g(T_{-t})} \delta d\theta \\ &\quad + \sum_{p \in \mathfrak{p}_{>\pi}} \int_0^{2\pi} \mathcal{O}(\delta^{2-2\mathfrak{a}+\epsilon}) \sigma(z) \delta d\theta \end{aligned}$$

but in the last term, since $\sigma = \mathcal{O}(\delta^{2(\alpha-1)})$, the integrand is $\mathcal{O}(\delta^{1+\epsilon})d\theta$ so

$$\int_{\partial\Sigma(\delta)} e_0 \left.\frac{d}{dt}\right|_{t=0} \langle \dot{u}_{-t}, \partial_{\nu_{-t}}\rangle_{g(T_{-t})} ds = \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \langle \dot{T}_t, \partial_{r_{-t}}\rangle_{g(T_{-t})} \delta d\theta + \mathcal{O}(\delta^{1+\epsilon}) \tag{6.55}$$

Thus as $\delta \to 0$ this approaches the final term in (6.51). Note that in the end $\ddot{\widetilde{u}}_0$ disappears completely. Next comes the middle term in (6.54). Using that $T_0 = id$, $\widetilde{u}_0 = id$, we have

$$\dot{e}_0 = \left.\frac{d}{dt}\right|_{t=0} e(T_t) + \left.\frac{d}{dt}\right|_{t=0} e(\widetilde{u}_t),$$

and by (3.1), $\frac{d}{dt}\big|_{t=0} e(\widetilde{u}_t) = \mathcal{O}(\delta^{1+\epsilon-2\alpha})$. Using this and $\ddot{T}_0 \equiv 0$ near $\mathfrak{p}$, we see that

$$\sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} -2\dot{e}_0 \langle \dot{T}_0, \partial_r\rangle_g \delta d\theta = \sum_{p\in\mathfrak{p}_{>\pi}} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} e(T_t)\langle \dot{T}_0, \partial_\nu\rangle_g + \mathcal{O}(\delta^\epsilon),$$

And this indeed approaches the middle term of (6.51). That the first term in (6.54) limits to the first term in (6.51) follows in the same way. This completes the proof ☐

6.6.2. *Zeros of the Hessian.* We can now prove

Corollary 6.12. *The C_j are conformal Killing fields.*

Proof. Assume that $u_0 = id$. For genus $\Sigma > 1$, $\mathcal{C}onf_0 = \{id\}$. From the decomposition (5.7) and our work above, it follows that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, G) = \left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, H_1) + \left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, H_2).$$

Note that this is a non-trivial statement, since it is by no means clear that, for example, the function $t \longmapsto E(u_t, g, H_1)$ is twice differentiable; but it indeed is, by exactly the same computations we just preformed. The fact that conformal maps are global minimizers of energy now implies that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, H_1) \geq 0,$$

so since the left hand side of our first equation is zero by Lemma 6.10, we arrive at

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(u_t, g, H_2) = 0.$$

But in fact $E(\cdot, g, H_2)$ if positive definite, as we now shoe. Cutting out conformal discs $D_j(\epsilon)$ as above, near $p \in \mathfrak{p}_{>\pi}$ the boundary term in (6.30) is

$$\begin{aligned}\lim_{\epsilon\to 0} \int_{\partial D_j(\epsilon)} \langle \nabla_{u_*\partial_\nu} C, C\rangle_{u^*H_2} ds &= \lim_{\epsilon\to 0} \int_0^{2\pi} \partial_\epsilon \langle C, C\rangle_{u^*H_2} \epsilon d\theta \\ &= \lim_{\epsilon\to 0} \int_0^{2\pi} \partial_\epsilon \left| w + \mathcal{O}^{2-2\alpha_j} \right| \epsilon d\theta \\ &= 0.\end{aligned}$$

Near $p \in \mathfrak{p}_{<\pi}$ the same computation shows that the contribution is also zero. We now see from (6.30) that

$$\int_M \left(\langle \nabla C, \nabla C \rangle_{u^* H_2} + \operatorname{tr}_g R^{H_2}(du, C, C, du) \right) dVol_g = 0,$$

which immediately implies that $C = 0$ by negative curvature. (In particular C is conformal Killing.)

Now assume the genus of Σ is 1. As above we lift to the universal cover and use (5.8). As in the previous paragraph we conclude that

$$\int_M \left(\langle \nabla C, \nabla C \rangle_{u^* K_2} + \operatorname{tr}_g R^{K_2}(du, C, C, du) \right) dVol_g = 0,$$

but now we can only have flatness and thus can only conclude that

$$^{K_2}\nabla C = 0. \tag{6.56}$$

The lift of K_2 to the universal cover, $\widetilde{K}_2$, written with respect to the global coordinates $\widetilde{z}$ on $\mathbb{C}$, is a constant coefficient metric. Hence (6.56) means $C \equiv v$ for some constant vector $v \in \mathbb{C}$, which, as desired, is conformal Killing. The first statement follows exactly as in the genus $\Sigma > 1$ case.

□

6.7. **Proof of Proposition 4.1.** Recall from Lemma 6.3 that the Fredholm map

$$L : r^{1-\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$$

satisfies

$$L(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}) \oplus \mathcal{K} = r^{1-\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}, \tag{6.57}$$

and the sum is L^2 orthogonal. This obviously implies

$$\left(L(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}) \oplus \mathcal{K}\right) \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}, \tag{6.58}$$

Since $L : r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \longrightarrow r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$ is also Fredholm for ϵ small, $L\left(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}\right) \subset L(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}) \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$ is a finite index inclusion, and we can find

$$W \subset L(r^{1-\epsilon}\mathcal{X}_b^{2,\gamma}) \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} \tag{6.59}$$

so that

$$\left(L\left(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}\right) + W\right) \oplus \mathcal{K} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{2,\gamma}. \tag{6.60}$$

We will now use the final paragraph in section 6.2, which says that vector fields like those in W which are sent via L to vector fields with higher vanishing rates than generically expected admit partial expansions. Precisely, any vector field $\psi \in W$ satisfies

$$\psi \in W \implies \begin{cases} \psi \in r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} \\ \psi = L\psi' \text{ for } \psi' \in r^{1-\epsilon}\mathcal{X}_b^{0,\gamma}, \end{cases} \tag{6.61}$$

so from (6.21) and (6.22) we have $\psi' = \psi_1 + \psi_2$, where

$$\psi_1 = \sum_{(s,p)\in\Lambda\cap[1-\epsilon,1+\epsilon]} r^s \log^p(r) a_{s,p}(\theta),$$

and $\psi_2 \in r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}$. Looking at the eigenvectors in section 6.4 shows that $\psi_1(z) = \lambda z$ for some $\lambda \in \mathbb{C}$. Thus $\psi' \in \mathrm{T}_{id}\mathcal{D} + r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}$, which of course implies

$$L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma}) + W = L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + \mathrm{T}_{id}\mathcal{D}). \tag{6.62}$$

Now by (6.60)

$$L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + \mathrm{T}_{id}\mathcal{D}) \oplus \mathcal{K} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}, \tag{6.63}$$

where $\mathcal{K} \perp L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} + \mathrm{T}_{id}\mathcal{D})$. Let

$$\pi_{\mathcal{K}} = \text{projection onto } \mathcal{K} \text{ in (6.63)}.$$

Now we add $\mathrm{T}_{id}\mathcal{T}_{>\pi}$ to the domain of L. Let $\psi \in \mathrm{T}_{id}\mathcal{T}_{>\pi}$ corresponding to $w \in \mathbb{C}^n$, so near $p_i \in \mathfrak{p}_{>\pi}$ we have $\psi \equiv w_i$ Since near p_i we have $Lw_i = 0$, we know that

$$L\psi \in r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}. \tag{6.64}$$

Using the basis for $\mathcal{K}$ in (6.44)–(6.45), (6.64) implies that we can write

$$\pi_{\mathcal{K}} L\psi = \sum_{j=1}^{N} \langle L\psi, J_{a^i}\rangle_{L^2} J_{a^i} + \sum_{k=1}^{M} \langle L\psi, C_k\rangle_{L^2} C_k,$$

for the L^2 inner product in (**??**). A simple integration by parts using shows that for $\widetilde{\psi} \in \mathcal{K}$ we have

$$\langle L\psi, \widetilde{\psi}\rangle_{L^2} = -4\pi\Re \sum_{p_i \in \mathfrak{p}_{>\pi}} w_i \operatorname{Res}|_{p_i} \Phi(\widetilde{\psi}).$$

This immediately implies that

$$\pi_{\mathcal{K}} L(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \oplus \mathrm{T}_{id}\mathcal{D} \oplus \mathrm{T}_{id}\mathcal{T}_{>\pi}) = \operatorname{span}\langle J_{a^i}\rangle,$$

and thus we have shown that

$$L\left(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \oplus \mathrm{T}_{id}\mathcal{D} \oplus \mathrm{T}_{id}\mathcal{T}_{>\pi}\right) \oplus \mathrm{T}\mathcal{C}on f_0 \cap r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma},$$

which is what we wanted.

6.8. **Proof of Proposition 4.2.** For the proof of Proposition 4.2 we will need a formula for the second variation of energy in the direction of an arbitrary $J_w \in \mathcal{J}_{\mathfrak{q}}$. First, we compute the first variation near a solution (HME($\mathfrak{q}$)).

Proposition 6.13. *Let (u_0, g_0, G) solve (HME($\mathfrak{q}$)) and satisfy Assumption 3.1. With notation as in the previous section, for any $w \in \mathbb{C}^{|\mathfrak{q}|}$ we have*

$$\left.\frac{d}{dt}\right|_{t=0} E(u_{tw}, g_0, G) = \Re\left(2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} \operatorname{Res}|_{p_i}\left(\iota_{J_w}\Phi(u_0)\right)\right), \tag{6.65}$$

and if $\Phi(u_0) = \phi_{u_0} dz^2$

$$\operatorname{Res}|_{p_i}\left(\iota_{J_w}\Phi\right) = w_i \operatorname{Res}|_{p_i}\phi_{u_0} \tag{6.66}$$

Corollary 6.14. *Let (u_0, g_0, G) be a solution to (HME($\mathfrak{q}$)). Then u_0 minimizes $E(\cdot, g_0, G)$ in its free homotopy class if and only if it solves (HME($\mathfrak{q}$)) with $\mathfrak{q} = \varnothing$, i.e. if and only if $Phi(u_0)$ is holomorphic on (Σ, g_0).*

Proof. That $\operatorname{Res}\Phi(u_0) = 0$ implies the minimizing property is the content of Proposition 5.2.

For the other direction, if (u_0, g_0, G) solves (HME($\mathfrak{q}$)) and satisfies Assumption 3.1, then

$$\operatorname{Res}\Phi(u) \neq 0 \implies u_0 \text{ is not energy minimizing}$$

since by (6.65), if $w \in \mathbb{C}^{|\mathfrak{q}|}$ has $\Re 2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} w_i \operatorname{Res}|_{p_i} \phi_{u_0} \neq 0$ (which is easy to arrange), then

$$\left.\frac{d}{dt}\right|_{t=0} E(u_{tw}, g_0, G) \neq 0,$$

contradicting minimality. □

Remark 6.15. *The one form $\iota_{J_w}\Phi(u_0)$ is not holomorphic (since J_w is not), but we will show that it still has a residue, meaning that the limit*

$$\operatorname{Res}\iota_{J_w}\Phi := \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{|z|=e}\iota_{J_w}\Phi(z)$$

exists.

By our assumption that $u_0 = id$ and decomposition (2.34), in conformal coordinates the metric G is expressed by

$$G = \sigma\,|dz|^2 + 2\Re\phi(z)dz^2. \tag{6.67}$$

Below, these coordinates are used on both the domain and the target.

Proof of Proposition 6.13. The proof is similar to the proof of Lemma 6.10. As always assume $u_0 = id$. We can write $u_{tw} = \widetilde{u}_{tw} \circ T_{tw}$ where $\widetilde{u}_{tw} \in \mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{D} \circ \mathcal{T}_{>\pi}$ and $T_{tw} \in \mathcal{T}_{\mathfrak{q}}$, and $\frac{d}{dt}\big|_{t=0} u_{tw} = J_w$ (see(4.7)). Then

$$\left.\frac{d}{dt}\right|_{t=0} E\left(u_{tw}, \Sigma\right) = \underbrace{\left.\frac{d}{dt}\right|_{t=0} E\left(u_{tw}, \Sigma - \bigcup_{i=1}^{|p|} D_{tw}(\delta)\right)}_{:=A(\delta)} + \underbrace{\left.\frac{d}{dt}\right|_{t=0} E\left(u_{tw}, \bigcup_{i=1}^{|p|} D_{tw}(\delta)\right)}_{:=B(\delta)}.$$

As in Lemma 6.10 $B(\delta) \to 0$ as $\delta \to 0$. Thus we arrive at

$$\left.\frac{d}{dt}\right|_{t=0} E\left(u_{tw}, \Sigma\right) = \lim_{\delta\to 0} A(\delta).$$

By the chain rule,

$$A(\delta) = \underbrace{\left.\frac{d}{dt}\right|_{t=0} E\left(u_{tw}, \Sigma - \bigcup_{\mathfrak{p}} D_0(\delta)\right)}_{:=A_1(\delta)} + \underbrace{\left.\frac{d}{dt}\right|_{t=0} E\left(u_0, \Sigma - \bigcup_{\mathfrak{p}} D_{tw}(\delta)\right)}_{:=A_2(\delta)} \tag{6.68}$$

As we will see shortly, the term $A_2(\delta)$ is not in general bounded as $\delta \to 0$, but we will show that $A_1(\delta)$ decomposes into a sum of two terms, $A_1(\delta) = A_1^1(\delta) + A_1^2(\delta)$ where $A_1^1(\delta) \sim -A_2(\delta)$ (i.e. it cancels the singularity), and $A_1^2(\delta)$ converges to the expression in (6.65). A_1 is an

integral over a smooth manifold with boundary so by the first variation formula (2.5), and in the last line using decomposition (6.67), if $\Sigma(\delta)$ is as in (6.53), we have

$$\begin{aligned} A_1 &= \int_{\partial\Sigma(\delta)} \langle J_w, \partial_\nu \rangle_G ds \\ &= \underbrace{-\sum_{p_i \in \mathfrak{p}_{<\pi}} \int_0^{2\pi} g\left(J_w, \partial_r\right) \delta d\theta}_{:=A_1^1} \underbrace{- \sum_{p_i \in \mathfrak{p}_{<\pi}} \int_0^{2\pi} \Re\Phi\left(J_w, \partial_r\right) \delta d\theta}_{A_1^2}. \end{aligned}$$

It thus remains only to show: 1) $\lim_{\delta\to 0} A_1^2 = \Re\left(2\pi i \operatorname{Res}\left(\iota_{J_w}\Phi\right)\right)$, 2) (6.66) holds, and 3) $\lim_{\delta\to 0}|A_1^1 + A_2| = 0$.

We prove numbers 1 and 2 together:

$$\int_{\partial D_0(\delta)} \Re\Phi\left(J_w, \partial_r\right) \delta d\theta = \int_{\partial D_0(\delta)} \Re\phi(z) dz^2 \left(J_w, \partial_r\right) \delta d\theta.$$

Since

$$dz^2\left(J_w, \partial_r\right) = J_w dz\left(\partial_r\right) = J_w \frac{z}{|z|},$$

we have

$$\int_0^{2\pi} \Re\Phi\left(J_w, \partial_r\right) \delta d\theta = \Re \int_0^{2\pi} J_w \phi(z) dz, \tag{6.69}$$

and by definition

$$\int_0^{2\pi} \Re\Phi\left(J_w, \partial_r\right) \delta d\theta = \Re \int_0^{2\pi} \iota_{J_w}\Phi(z). \tag{6.70}$$

So

$$\begin{aligned} \sum_{p_i \in \mathfrak{p}_{<\pi}} \lim_{\delta\to 0} \int_0^{2\pi} J_w \phi(z) dz &= \sum_{p_i \in \mathfrak{p}_{<\pi}} \lim_{\delta\to 0} \int_0^{2\pi} \left(-w_i + \mathcal{O}(|z|)\right) \phi(z) dz \\ &= -2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} w_i \operatorname{Res}|_{p_i} \phi_{u_0}. \end{aligned} \tag{6.71}$$

Putting (6.71) together with (6.70) gives us what we wanted.

For number 3, note that from the expression $g = e^{2\mu}|z|^{2(\alpha-1)}$ we have

$$A_1^1(\delta) = \int_0^{2\pi} \langle -w_i + \mathcal{O}(|z|), \partial_r \rangle_g r d\theta = \int_0^{2\pi} \langle -w_i, \partial_r \rangle_g r d\theta + \mathcal{O}(\delta^{2\alpha})$$

Using (6.50) and that fact that near $p \in \mathfrak{p}$, $T_{tw}^{-1}(z) = z + tw$ parametrizes the boundary of $\partial D_{tw}(\delta)$ we have

$$A_2 = \sum_{i=1}^{|p|} \int_0^{2\pi} \langle \dot{T}_0^{-1}, \partial_r \rangle_g r d\theta = \sum_{i=1}^{|p|} \int_0^{2\pi} \langle w_i, \partial_r \rangle_g ds + \mathcal{O}(\delta^{2\alpha}).$$

Thus $|A_1^1 + A_2| = \mathcal{O}(\delta^{2\alpha})$, and the proof is finished. $\square$

Now suppose that u_0 solves (HME($\mathfrak{q}$)). Thus u_0 is an absolute minimum of $E(\cdot, g, G)$. By differentiating again, we get the following as a corollary to Proposition 6.13.

Corollary 6.16. *Suppose that* (u_0, g, G) *solves* (HME($\mathfrak{q}$)) *and satisfies Assumption 3.1. Then*

$$\frac{d^2}{dt^2}\Big|_{t=0} \widetilde{E}(u_{tw}) = \Re\left(2\pi i \sum_{p_i \in \mathfrak{p}_{<\pi}} \operatorname{Res}|_{p_i}\left(\iota_{J_w}\Phi(J_w)\right)\right) = \Re\left(2\pi i \sum_{p_i\in\mathfrak{p}_{<\pi}} w_i \operatorname{Res}|_{p_i}\phi(J_w)\right) \quad (6.72)$$

where $\Phi(J_w)dz^2 = \frac{d}{dt}\big|_{t=0}\phi_{u_{tw}}dz^2$

We now conclude the proof Proposition 4.2 using Proposition 6.13

Proof of Proposition 4.2. We proceed by interpreting the formula for the Hessian in Corollary 6.16 in light of the characterization of the Hessian of the energy functional at a solution to (HME($\mathfrak{q}$)) in Corollary 6.12. By Corollary 6.12, the Hessian of the energy functional is positive definite on any compliment of the conformal Killing fields, while by Corollary 6.14, if $\operatorname{Res} J_w = 0$, J_w is a zero of the Hessian. If genus $\Sigma > 1$, there are no conformal Killing fields, so the map (4.8) is an injective linear map between vector spaces of the same dimension, thus an isomorphism. If genus $\Sigma \leq 1$, any space $\mathcal{H}arm'_{\mathfrak{q}} \subset \mathcal{H}arm_{\mathfrak{q}}$ with tangent space complimentary to $\mathcal{C}onf_0$ will satisfy (4.10).

□

7. Closedness: limits of harmonic diffeomorphisms

In this section we prove that $\mathcal{H}(\mathfrak{q})$ (see Proposition 3.5) is closed. In fact we prove the stronger result

Theorem 7.1. *Let* (u_k, g_k, G_k) *be a sequence of solutions to (HME($\mathfrak{q}$)) where* $g_k \to g_0$ *with* $g_k \in \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$, $G_k \to G_0$ *with* $G_k \in \mathcal{M}^*_{2,\gamma,\nu}(G_0, \mathfrak{p}, \mathfrak{a})$, *and all the* u_k *are in Form 2.2,. Assume that* $\kappa_{G_k} \leq 0$ *and that each* u_k *has non-vanishing Jacobian away from* $u_k^{-1}(\mathfrak{p})$. *Then the* u_k *converge to a map* u_0 *so that* (u_0, g_0, G_0) *solves (HME($\mathfrak{q}$)) in Form 2.2. The convergence is* $C^{2,\gamma}_{loc}$ *away from* $u_0^{-1}(\mathfrak{p})$. *For a precise description of the convergence near the cone points, see Corollary 7.9 below.*

To be precise about the convergence of the G_k, in conformal coordinates near $p \in \mathfrak{p}$ we have

$$G_k = c_k e^{2\mu_k} |w|^{2(\alpha-1)} |dw|^2,$$

and

$$G_0 = c_0 e^{2\mu_0} |w|^{2(\alpha-1)} |dw|^2.$$

By $G_k \to G_0$, we mean that for some $\sigma > 0$,

$$\begin{aligned} &\mu_k \to \mu \text{ in } r^\nu C^{2,\gamma}_b(D(\sigma)) \\ &c_k \to c_0 \end{aligned} \quad (7.1)$$

(The b-Hölder spaces are defined in (2.23)–(2.25)). Away from the cone points $G_k \to G$ in $C^{2,\gamma}_{loc}$. We can (and do) reduce to the case $c_k = c_0 = 1$ by replacing G_k by G_k/c_k and G_0 by G_0/c_0. Note that (7.1) easily implies that near each cone point the scalar curvature functions κ_{G_k} converge to κ_{G_0} in $C^{0,\gamma}_b$ (see (2.26)); in particular are they uniformly bounded in absolute value.

For the g_k, we make the stronger assumption that near $u^{-1}(\mathfrak{p})$ the metrics look like the standard round conic metric g_α (see (2.8)). To do this uniformly, we need the uniform

bound on the modulus of continuity obtained in the next section; the precise statement of this assumption is in section 7.2. In the end the theorem is true as stated (i.e. without this stronger assumption), since we will change the domain metric in a bounded way and in its conformal class.

We refer the reader to the introduction for an outline of the subsequent arguments. Before we prove Theorem 7.1, we use it to prove

Proposition 7.2. *$\mathcal{H}(\mathfrak{q})$ is closed.*

Proof. Let $t_k \in \mathcal{H}(\mathfrak{q})$ be a sequence such that $t_k \to t_0$, and let $\mathfrak{c}_k$ be the corresponding conformal structures, so $\mathfrak{c}_k \to \mathfrak{c}_0$. We again uniformize locally, i.e. we choose diffeomorphisms v_k so that $id : (\Sigma, \mathfrak{c}_0) \longrightarrow (\Sigma, v_k^* \mathfrak{c}_k)$ is conformal near $\mathfrak{p}$ in such a way that $v_k \to id$ in C^∞. By assumption, there is a rel. $\mathfrak{q}$ minimizer $u_k : (\Sigma, \mathfrak{c}_k) \longrightarrow (\Sigma, G)$. Let g_k be metrics in $\mathfrak{c}_k$ and g_0 be a metric in $\mathfrak{c}_0$ that are conic near near $u_k^{-1}(\mathfrak{p})$ with cone angles $\mathfrak{a}$. Then $u_k \circ v_k : (\Sigma, v_k^* g_k) \longrightarrow (\Sigma, G)$ are also rel. $\mathfrak{q}$ minimizers, but now $v_k^* g_k \in \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a})$, and thus Theorem 7.1 applies. The limiting map u_0 is the minimizer we desire, to $\mathcal{H}(\mathfrak{q})$ so closed. $\square$

7.1. **Energy bounds and uniform continuity.**

The u_k in Theorem 7.1 are in fact a rel. $\mathfrak{q}$ energy minimizing sequence for the metric G_0, meaning that

$$\limsup_{k\to\infty} E(u_k, g_k, G_0) = \inf_{u \sim_{rel.\mathfrak{q}} id} E(u, g_k, G_0). \tag{7.2}$$

This follows from Proposition 5.2, since for any $u \sim_{rel.\mathfrak{q}} id$

$$\begin{aligned}\limsup_{k\to\infty} E(u_k, g, G_0) &= \limsup_{k\to\infty} E(u_k, g_k, G_k) \\ &\leq \lim_k E(u, g_k, G_k) \\ &= E(u, g_0, G_0).\end{aligned}$$

Assume that genus $\Sigma > 0$. It is standard that the u_k are an equicontinuous sequence. Thus they subconverge. Let

$$R := u_0^{-1}(\Sigma_{\mathfrak{p}}).$$

On this regular set, it again standard that the u_k converge uniformly in $C^{2,\gamma}_{loc}$ on compact sets of R. [Tr], [J].

For $\Sigma = S^2$, lift as in (5.9) to a branched cover and applying the above arguments gives the same results. To summarize, we have

Lemma 7.3. *The u_k converge in $C^0(\Sigma) \cap C^{2,\gamma}_{loc}(R)$ to a map u_0. On each compact set of R the Jacobian of u_0 is bounded below by a positive constant.*

7.2. **Upper bound for the energy density and lower bound on the Jacobian.**

Fix $p \in \mathfrak{p}$, and let

$$q_k = u_k^{-1}(p),$$

we can pass to a subsequence so that

$$q_k \to q_0,$$

for some q_0. Let $S \subset \Sigma$ be any set containing $u_0^{-1}(p)$ so that $S \cap u_0^{-1}(\mathfrak{p} - \{p\}) = 0$ and S is diffeomorphic to a disc, and choose conformal maps

$$\begin{aligned} F_k : D &\longrightarrow S \\ 0 &\longmapsto q_k \end{aligned}$$

so that $F_k \to F_0$ in C^∞. Finally, define

$$w_k := u_k \circ F_k.$$

Thus we have a sequence of harmonic maps $w_k : D \longrightarrow (\Sigma, G_k)$ with $w_k(0) = p$. By uniform continuity, we may choose a single conformal coordinate chart containing $w_k(D) = u_k \circ F_k(D)$. By abuse of notation, we denote these coordinates by w. Our goal is to prove uniform estimates for the w_k. Specifically, we wish to control their energy densities and Jacobians. The remarkable fact here is that, although being harmonic is conformally invariant, if the domain is given a conic metric with cone points at the inverse images of the cone points in the target, then uniform control, from above and below, in terms of the energy is available exactly when the metric on the domain has cone angles equal to that of the corresponding cone points in the target metric. Consider the maps

$$w_k : (D, g_\alpha) \longrightarrow (\Sigma, G_k), \tag{7.3}$$

where $2\pi\alpha$ is the cone angle at p and again $g_\alpha = |z|^{2(\alpha-1)} |dz|^2$.

Proposition 7.4. *The maps (7.3) have uniformly bounded energy density (see (3.1)), i.e. for some $C > 0$*

$$e_k(z) = e(w_k, g_\alpha, G_k)(z) < C \text{ for } |z| \leq 1/2. \tag{7.4}$$

At $z = 0$ we have the lower bound

$$0 < c \leq \lim_{z \to 0} e_k(z, g_\alpha, G_k) \tag{7.5}$$

for some uniform c.

Before we begin the proof, we remark that by Form 2.2,

$$\begin{aligned} w_k(z) &= \lambda_k z + v_k(z) \\ v_k &\in r^{1+\epsilon} C_b^{2,\gamma}. \end{aligned} \tag{7.6}$$

As we will show below

$$\lambda_k^{2\alpha} = \lim_{z \to 0} e_k(z, g_\alpha, G_k), \tag{7.7}$$

so showing (7.5) reduces to showing that $0 < c \leq \lambda_k$ for some uniform c.

Proof of Proposition 7.4. For any harmonic map $w : (D, \sigma |dz|^2) \longrightarrow (D, \rho |dw|^2)$, as in [W3], let

$$h(z) = \frac{\rho(w(z))}{\sigma(z)} |\partial_z w|^2, \qquad \ell(z) = \frac{\rho(w(z))}{\sigma(z)} |\partial_{\overline{z}} w|^2, \tag{7.8}$$

so the energy density and the Jacobian satisfy, respectively

$$e = h + \ell \qquad \& \qquad J = h - \ell. \tag{7.9}$$

The lemma follows from analysis of the following inequality and identity, which are standard and can be found for example in [SY], where they appear as equations (1.19) and (1.17), respectively.

$$\begin{aligned} \Delta e(u) &\geq -2\kappa_\rho J + 2\kappa_\sigma e(u) \\ \Delta \log h &= -2\kappa_\rho \left(h - \ell\right) + \kappa_\sigma . \end{aligned} \tag{7.10}$$

Here κ_ρ and κ_σ are the scalar curvature functions for the range and domain, respectively, and Δ is the Laplacian for $\sigma \left|dz\right|^2$. The second equation holds only when $h(z) \neq 0$, and of course both equations make sense only when σ and ρ are sufficiently regular. For the w_k in (7.3), the equations simplify as follows: 1) $\sigma(z) \left|dz\right|^2 = g_\alpha$ has $\kappa_\sigma \equiv 0$ away from $z = 0$, 2) $\kappa_{\rho_k} \leq 0$, $J \geq 0$ by the assumptions of Theorem (7.1), and 3) $\kappa_{\rho_k} > -C$ by the remark about the convergence of scalar curvature at the beginning of the section. Therefore, we restrict our attention to the inequalities

$$\Delta e_k \geq 0 \tag{7.11}$$

$$\Delta \log h_k \leq C h_k . \tag{7.12}$$

To prove (7.4), we use (7.11) as follows. Since $\Delta = \left|z\right|^{2(1-\alpha)} \Delta_0$ where $\Delta_0 = 4\partial_z \partial_{\overline{z}}$ is the euclidean Laplacian, we also have

$$\Delta_0 e_k \geq 0, \tag{7.13}$$

away from $z = 0$. We claim that in fact each e_k is a subsolution to (7.13) on all of D. To see that this is true, write $e_k = h_k + \ell_k$ as in (7.9). Using (7.6), we have

$$\begin{aligned} h_k(z) &= \frac{\rho_k(w_k(z))}{\left|z\right|^{2(\alpha-1)}} \left|\partial_z w_k(z)\right|^2 \\ &= \lambda_k^{2(\alpha-1)} \left|1 + \left|z\right|^{2(1-\alpha)} v_k(z)\right|^{2(\alpha-1)} e^{2\mu_k(w_k(z))} \left|\lambda_k + \partial_z v_k(z)\right|^2 \\ &= \lambda_k^{2\alpha} \left|1 + \left|z\right|^{2(1-\alpha)} v_k(z)\right|^{2(\alpha-1)} e^{2\mu_k(w_k(z))} \left|1 + \partial_z v_k(z)/\lambda_k\right|^2 , \end{aligned}$$

so since $v_k \in r^{1+\epsilon} C_b^{2,\gamma}$ and $\mu_k \in r^\nu C_b^{2,\gamma}$, we have

$$h_k - \lambda_k^{2\alpha} \in r^\epsilon C_b^{1,\gamma} \tag{7.14}$$

for some $\epsilon > 0$. Similarly, using $\partial_{\overline{z}} z = 0$,

$$\ell_k \in r^\epsilon C_b^{1,\gamma}. \tag{7.15}$$

Therefore

$$e_k = \lambda_k^{2\alpha} + f_k(z), \tag{7.16}$$

where $f_k \in r^\epsilon C_b^{1,\gamma}(D)$. In particular, $r\partial_r e_k \to 0$ as $r \to 0$. Therefore, for any non-negative function $\zeta \in C_c^\infty(D)$.

$$\begin{aligned} \int_D \nabla e_k \cdot \nabla \zeta dxdy &= - \lim_{\epsilon \to 0} \int_{D - D(\epsilon)} (\Delta e_k) \zeta dxdy - \int_{r=\epsilon} (\partial_r e_k) \zeta r d\theta \\ &\leq 0, \end{aligned} \tag{7.17}$$

so the e_k are indeed subsolutions. By (7.16), each e_k is a bounded function, so the standard theory of subsolutions to elliptic linear equations (see section 5 of [Mo]) now implies that for some uniform constant $C > 0$,

$$\sup_{z \in D(1/2)} e_k(z) \leq C \int_D e_k dxdy.$$

The right hand side is controlled by the energy,

$$\begin{aligned} C \int_D e_k dxdy &= C \int_D e_k \frac{|z|^{2(\alpha-1)}}{|z|^{2(\alpha-1)}} dxdy \\ &\leq C \int_D e_k |z|^{2(\alpha-1)} dxdy \\ &= CE(w_k, D, G_k) \\ &\leq C', \end{aligned}$$

and this establishes (7.4).

It remains to prove (7.5). From (7.14) and (7.15), we now see that

$$\lim_{z \to 0} e_k(z) = \lim_{z \to 0} h_k(z) = \lambda_k^{2\alpha}, \tag{7.18}$$

Thus (7.7) is correct, and to prove (7.5), it is equivalent to prove that $c < \lambda_k$ for some uniform c.

To do so, we use (7.12). Dropping the k's for the moment, by (7.14) and the fact that the logarithm is smooth and vanishes simply at 1, we have

$$\log h - \log |\lambda|^{2\alpha} \in r^\epsilon C_b^{2,\gamma}. \tag{7.19}$$

We will now apply the assumption from Theorem 7.1 regarding the Jacobian, specifically that $J = h - \ell > 0$ on compact sets away from 0, and thus by continuity and (7.18) we may choose a δ satisfying

$$0 < \delta \leq \frac{1}{2} \inf_{z \in D} h(z).$$

Thus h/δ is bounded from below by 2. We also need control from above. We already know by (7.4) that $h + \ell = e < c$ for some $c > 0$, so $\sup_{z \in D} h(z) < c$ for some constant depending only on the energy. Using this and the obvious bound $h \leq \frac{c}{\log(c'/\delta)} \log\left(\frac{h}{\delta}\right)$ we conclude from (7.12) that

$$\Delta \log (h/\delta) \leq \frac{c}{\log(c'/\delta)} \log\left(\frac{h}{\delta}\right). \tag{7.20}$$

The upshot is that $\log h/\delta \geq \ln 2 > 0$, so this inequality looks promising for an application of the Harnack inequality. We use the following explicit inequality, inspired by Lemma 6 of [He].

Remark 7.5. *The lemma we are about to prove has slightly more general assumptions than seem necessary, but they are necessary for the analogous statement in section 8 where we deal with the case $\mathfrak{p}_{=\pi} \neq \varnothing$.*

Lemma 7.6 (Harnack Inequality)**.** *Let Δ denote the Laplacian on the standard cone (D, g_α). Let $f : D \longrightarrow \mathbb{R}$, $f \in C^2(\overline{D} - \{0\})$, $f > 0$, and assume that for some $\sigma > 0$,*

$$(\Delta - \sigma) f \leq 0 \text{ on } D - \{0\} .$$

Assume furthermore that

$$\begin{aligned} &f = a + b(\theta) + v(r, \theta) \\ &a \in \mathbb{C} \\ &b \in C^\infty(S^1) \\ &v \in r^\epsilon C_b^{2,\gamma}(D) \end{aligned} \tag{7.21}$$

Then if $\sigma < \epsilon^2$,

$$\liminf_{z \to 0} f = a + \inf b \geq e^{-c\sigma} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta \tag{7.22}$$

for some $c > 0$.

Before proving the lemma, we conclude the proof of (7.5) (and thus of Proposition 7.4) by applying the lemma to (7.20) as follows. By (7.19), the lemma applies to (7.20) with $f = \log h/\delta$ and $b \equiv 0$. We will choose $\eta > 0$ small so that if

$$\delta_k = \min \left\{ \eta, \frac{1}{2} \inf_{z \in D - 0} h_k(z) \right\} \tag{7.23}$$

then the hypotheses of the lemma are satisfied.

Thus by the lemma

$$\begin{aligned} \lim_{z \to 0} \log(h_k(z)/\delta_k) &\geq e^{\frac{c}{\log(c'/\delta_k)}} \frac{1}{2\pi} \int_0^{2\pi} \log(h_k/\delta_k) d\theta \\ \lim_{z \to 0} \log h_k(z) &\geq e^{\frac{c}{\log(c'/\delta_k)}} \left(\int_0^{2\pi} \log h_k \; d\theta \right) - \left(e^{\frac{c}{\log(c'/\delta_k)}} - 1 \right) \log \delta_k . \end{aligned}$$

It is trivial to check that $\left(e^{\frac{c}{\log(c'/\delta_k)}} - 1 \right) \log \delta_k > c''$ for some c'' independent of δ_k. Thus, since $e^{\frac{c}{\log(c'/\delta_k)}}$ is bounded above, for some c, we have

$$\lim_{z \to 0} \log h_k(z) = |\lambda_k|^{2\alpha} \geq -c \left| \int_0^{2\pi} \log h_k(e^{i\theta}) \; d\theta \right| + c \tag{7.24}$$

By Lemma 7.3, the Jacobians, J_k, and thus the h_k, are uniformly bounded below on ∂D. Thus (7.24) gives a uniform lower bound for $h_k(0)$ from below, which finishes the proof of Proposition 7.4 modulo the proof of Lemma 7.6.

□

Proof of Lemma 7.6. To prove the lemma we will work in normal polar coordinates. Let $\phi = \theta, \rho = r^\alpha/\alpha$. In these coordinates $g_\alpha = d\rho^2 + \alpha^2 \rho^2 d\phi^2$ and $\Delta = \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\alpha^2\rho^2}\partial_\phi^2$. The fact that $f - (a + b(\theta)) \in r^\epsilon C_b^{2,\gamma}$ implies that $f - (a + b(\theta)) \in \rho^{\epsilon/\alpha} C_b^{2,\gamma}$. The exact vanishing rate is irrelevant, and we refer to it henceforth as ϵ.

The proof proceeds by comparing f to the solution $\widetilde{f}$ to the equation

$$\begin{aligned} (\Delta - \sigma)\widetilde{f} &= 0 \\ \widetilde{f}|_{\partial D} &= f|_{\partial D} \end{aligned} \tag{7.25}$$

That such a solution exists follows, for example, from separation of variables from [T], in particular there is a solution that is continuous on $\overline{D}$ with the asymptotic behavior

$$\begin{aligned} \widetilde{f}(z) - \widetilde{f}(0) &\in r^a C^{2,\gamma}_b \\ a &= \min\{2\alpha, 1\}. \end{aligned} \tag{7.26}$$

It follows that $F = f - \widetilde{f}$ satisfies

$$\begin{aligned} (\Delta - \sigma) F &\le 0 \text{ away from } z = 0 \\ F|_{\partial D} &= 0, \text{ where } \partial D = \{\rho = 1/\alpha\}. \end{aligned} \tag{7.27}$$

The main technical challenge is to show that $F(\rho, \phi) \ge 0$. Assuming this for the moment, we have in particular that

$$a + b(\theta) \ge \widetilde{f}(0). \tag{7.28}$$

Let

$$\gamma(\rho) := \frac{1}{2\pi} \int_0^{2\pi} \widetilde{f}(\rho, \phi) d\phi.$$

Then $\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \sigma\right)\gamma = 0$, so since $\gamma(0) = \widetilde{f}(0)$,

$$\gamma(\rho) = \widetilde{f}(0) \sum \frac{\sigma^m \rho^{2m}}{2^{2m}(m!)^2}.$$

But then by (7.28) and $F|_{\partial D} = 0$, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} f(1/\alpha, \phi) d\phi &= \int_0^{2\pi} \widetilde{f}(1/\alpha, \phi) d\phi \\ &= \widetilde{f}(0) \sum \frac{\sigma^m (1/\alpha)^{2m}}{2^{2m}(m!)^2} \\ &\le e^{\sigma/4\alpha^2} (a + b(\theta)), \end{aligned}$$

which is equivalent to (7.22).

It remains to show that $F \ge 0$. Switch back to conformal coordinates, $z = re^{i\theta}$. By assumption (7.21), $F(z) = F(r, \theta)$ is continuous on $[0, 1]_r \times S^1_\theta$, thus attains a minimum. If that minimum is on $r = 1$, we are done by (7.27). If it is in the interior and away from $z = 0$, say at z_0, then (7.27) implies that $0 \le \Delta F \le \sigma F$, so since $\sigma \ge 0$, $F(z_0) \ge 0$.

Finally, assume that F attains its minimum on $r = 0$, and suppose that

$$I := \inf_{r=0} F(r, \theta) < 0$$

We will perturb F to a function F_μ solving $(\Delta - \sigma) F_\mu \le 0$ with a negative interior minimum. This is a contradiction by the previous paragraph, so $I \ge 0$ and thus $F \ge 0$.

Consider the function

$$F_\mu(z) = F(z) - \mu |z|^\nu, \tag{7.29}$$

where $\mu > 0, \nu \geq \sqrt{\sigma}$. Since $(\Delta - \sigma)\,|z|^{\nu} = |z|^{\nu}\left(|z|^{-2\alpha}\nu^2 - \sigma\right) \geq 0$, F_μ also satisfies $(\Delta - \sigma)\,F_\mu(z) \leq 0$. Since $F_\mu|_{\partial D} \equiv -\mu$ and $\liminf_{z\to 0} F_\mu(z) = I$, if $I \leq -\mu$ then $F_\mu = F_\mu(r,\theta)$ has a (negative) minimum away from ∂D. On the other hand, given any (r,θ) with $r > 0$,

$$F_\mu(0,\theta) \geq F_\mu(r,\theta) \iff \frac{F(0,\theta) - F(r,\theta)}{r^\nu} \geq -\mu.$$

Thus *if* we can find $\mu > 0$ so that for some $r > 0, \theta, \nu$

$$I \leq -\mu \leq \frac{F(0,\theta) - F(r,\theta)}{r^\nu}, \tag{7.30}$$

then F_μ will have a negative interior minimum. By the assumptions of the lemma and (7.26) we have that $F(0,\theta) - F(r,\theta) \in r^\epsilon C_b^{2,\gamma}$, and in particular $\frac{|F(0,\theta)-F(r,\theta)|}{r^\epsilon} < C$. Thus, as long as $\nu < \epsilon$, we have that

$$\frac{|F(0,\theta) - F(r,\theta)|}{r^\nu} \to 0 \text{ as } z \to 0.$$

Thus, taking the second line in (7.29) into account, (7.30) can be attained if

$$\sqrt{\sigma} < \epsilon. \tag{7.31}$$

This finishes the proof. $\square$

7.3. **Harmonic maps of C_α.** Now we prove a classification lemma for harmonic maps of the standard cone C_α defined in section 2.1.

Lemma 7.7. *Let $u : C_\alpha \longrightarrow C_\alpha$ be a smooth harmonic map (i.e. a solution to (HME($\mathfrak{q}$))) in Form 2.2, with uniformly bounded energy density, that is the uniform limit of a sequence of orientation-preserving homeomorphisms fixing the cone point. Then u is a dilation composed with a rotation.*

Proof. Let e be the energy density function of u, and let h, ℓ, and J be the functions defined in (7.8). We will use a differential inequality for ℓ, similar to those used in the proof of Proposition 7.4. Namely, (1.18) from [SY] gives

$$\Delta\ell \geq 2\kappa_\rho J\ell + 2\kappa_\sigma\ell,$$

where κ_ρ and κ_σ are the scalar curvatures on the domain and target, respectively, and the inequality holds only away from $z = 0$. Since C_α is flat away from the cone point, this gives $\Delta\ell(z) \geq 0$ when $z \neq 0$. The assumption that the energy density of u is uniformly bounded implies ℓ is uniformly bounded ($\ell < e$), and on the other hand, as in (7.17), ℓ is a subsolution on all of $\mathbb{C}$. Thus ℓ is identically constant. (There are no non-constant, bounded entire subsolutions.) Since $\ell \in r^\epsilon C_b^{2,\gamma}$, $\lim_{z\to 0}\ell = 0$, and thus $\ell \equiv 0$, i.e. $\partial_{\overline{z}} u = 0$ when $z \neq 0$ Since u bounded near 0, it is in fact entire. The fact that u is the uniform limit of homeomorphisms and takes Form 2.2 implies that it is $1-1$ on the inverse image of an open ball around 0. Since the only entire holomorphic function with this behavior that fixes 0 is $u(z) = az$ for $a \in \mathbb{C}^*$, the proof is complete. $\square$

7.4. **Uniform $C_b^{2,\gamma}$ bounds near $\mathfrak{p}$ and the preservation of Form 2.2.** We will now make precise the sense in which the w_k in (7.3) are uniformly bounded near $\mathfrak{p}$. Let $D(R) = \{z \in C : |z| < R\}$. We will need the well-known Rellich lemma for b-Hölder spaces; given $0 < \gamma' < \gamma < 1$, $c' < c$, and non-negative integers $k' \leq k$ the containment

$$r^c C_b^{k,\gamma}(D(R)) \subset r^{c'} C_b^{k',\gamma'}(D(R)) \tag{7.32}$$

is compact. Given a smooth function $f : \mathbb{C} \longrightarrow \mathbb{R}$, define

$$\|f\|_{c,k,\gamma,R} := \|f\|_{r^c C_b^{k,\gamma}(D(R))}, \tag{7.33}$$

And for any map $w : D(R) \longrightarrow D(R)$ with

$$w(z) = \lambda z + v(z) \tag{7.34}$$

and $v \in r^c C_b^{2,\gamma}(D(R))$, let

$$\boxed{[w]_{c,k,\gamma,R} := \|v\|_{c,k,\gamma,R}} \tag{7.35}$$

Having established the uniform energy density bound, we now set out to prove

Proposition 7.8. *For the w_k in (7.3), there exist uniform constants $\sigma, C, \epsilon > 0$ such that*

$$[w_k]_{1+\epsilon,2,\gamma,\sigma} < C. \tag{7.36}$$

From this, Proposition 7.4, and the fact that the containment (7.32) is compact, we immediately deduce

Corollary 7.9. *The $w_k = \lambda_k z + v_k$ converge to a map $w_0 = \lambda_0 z + v_0$, in the sense that*

$$\begin{aligned} &\lambda_k \to \lambda_0 \\ &v_k \to v_0 \ \text{in } r^{1+\epsilon} C_b^{2,\gamma}, \end{aligned}$$

for some $\epsilon, \gamma > 0$. Since the F_k in $u_k \circ F_k = w_k$ converge in C^∞ to some univalent conformal map F_0, the limit in $u_k \to u_0$ is of Form 2.2.

Before we prove Proposition 7.8, we discuss scaling properties of the norm in (7.33). Let $R > 0$, $\sigma > 0$, and let f be a function defined on $D(R)$. Writing

$$f_\sigma(z) = f(\sigma z),$$

it is trivial to verify from (2.23)–(2.25) that

$$\|f_\sigma\|_{c,k,\gamma,R/\sigma} = \sigma^c \|f\|_{c,k,\gamma,R} \tag{7.37}$$

For $w : D \longrightarrow D$ as in (7.34), we may define, locally

$$\frac{1}{\tau} w_\sigma(z) = \frac{\lambda\sigma}{\tau} z + \frac{v_\sigma(z)}{\tau}. \tag{7.38}$$

By (7.37) we have

Lemma 7.10. *If $w(z) = \lambda z + v(z)$, then*

$$\left[\frac{1}{\sigma} w_\tau\right]_{c,k,\gamma,R/\tau} = \frac{\tau^c}{\sigma} [w]_{c,k,\gamma,R}$$

We will apply this lemma directly to the w_k in (7.3). For these maps $G_k = e^{2\mu_k} |w|^{2(\alpha-1)} |dw|^2$, and the map $\frac{1}{\sigma}(w_k)_\tau(z)$ is an expression in normalized conformal coordinates (see (2.19)) of the map

$$w_k : (D, g_\alpha/\tau^{2\alpha}) \longrightarrow (\Sigma, G_k/\sigma^{2\alpha}), \tag{7.39}$$

meaning simply that since

$$g_\alpha/\tau^{2\alpha} = \frac{|z|^{2(\alpha-1)}}{\tau^{2\alpha}} |dz|^2 \quad \& \quad G_k/\sigma^{2\alpha} = e^{2\mu_k(w)} \frac{|w|^{2(\alpha-1)}}{\sigma^{2\alpha}} |dw|^2 ,$$

doing $z \mapsto \tau z$ and $w \mapsto \sigma w$ gives

$$g_\alpha/\tau^{2\alpha} = |z|^{2(\alpha-1)} |dz|^2 \quad \& \quad G_k/\sigma^{2\alpha} = e^{2\mu_k(\sigma w)} |w|^{2(\alpha-1)} |dw|^2 .$$

Note that it follows immediately from (3.5) that (7.39) is harmonic. Before we begin the proof of Proposition 7.8, we recall that by (7.5) we know that $\lambda_k \to \lambda_0$ for some $\lambda_0 > 0$, and, by setting $G_k = G_k/\lambda_k^{2\alpha}$, we may assume without loss of generality that

$$\lambda_k \equiv 1$$

so that the normalized conformal coordinate expression is $w_k = z + v_k(z)$.

Proof of Proposition 7.8. We proceed by contradiction. Supposing Proposition 7.8 false, we will produce a sequence $\sigma_k \to 0$ so that the scaled maps

$$\frac{1}{\sigma_k} w_{k,\sigma_k},$$

converge to a harmonic map $w_\infty : C_\alpha \longrightarrow C_\alpha$ satisfying the assumptions but not the conclusion of Lemma 7.7. This contradiction proves the lemma.

If (7.36) does not hold then for all $\sigma, C > 0$ there is a k such that

$$[w_k]_{1+\epsilon,2,\gamma,\sigma} > C. \tag{7.40}$$

Thus, for every $l \in \mathbb{N}$ there is a k_l such that $[w_{k_l}]_{1+\epsilon,2,\gamma,1/l} > l^\epsilon$, and passing to a subsequence, we assume that $[w_k]_{1+\epsilon,2,\gamma,1/k} > k^\epsilon$. Since the L.H.S. is monotone decreasing in the radius, for each k there is a number $\sigma_k < 1/k$ such that $[w_k]_{1+\epsilon,2,\gamma,\sigma_k} = \sigma_k^{-\epsilon}$. or, by the scaling properties in Lemma 7.10

$$\boxed{\left[\frac{1}{\sigma_k} w_{k,\sigma_k}\right]_{1+\epsilon,2,\gamma,1} = 1} \tag{7.41}$$

By the remarks immediately preceding the proof, this map, which is the normalized coordinate expression of the map

$$w_k : (D, g/\sigma_k^{2\alpha}) \longrightarrow (\Sigma, G_k/\sigma_k^{2\alpha}) \tag{7.42}$$

is harmonic and in normalized conformal coordinates satisfies $\frac{1}{\sigma_k} w_{k,\sigma_k}(z) = z + \frac{v_{k,\sigma_k}(z)}{\sigma_k}$. Set

$$\widetilde{v}_k := \frac{v_{k,\sigma_k}(z)}{\sigma_k} \tag{7.43}$$

By the obvious identity for the energy density, $e(w_k, g/\sigma_k^2, G_k/\sigma_k^2) = e(w_k, g, G_k)$, the uniform bound (7.4) holds for the maps in (7.42), and thus they converge on compact subsets of $(D, g_\alpha/\sigma_k^{2\alpha}) = (D(1/\sigma_k), g_\alpha)$ to a map $w_\infty : C_\alpha \longrightarrow C_\alpha$. The following claim will finish the proof of Proposition 7.8 since it contradicts Lemma 7.7.

Claim 7.11. *The map w_∞ is in Form 2.2, with*

$$w_\infty(z) = z + v_\infty(z) \tag{7.44}$$

and $\widetilde{v}_k \to v_\infty \in r^{1+\epsilon} C_b^{2,\gamma}(D)$, for $\epsilon', \gamma' > 0$ sufficiently small. Furthermore

$$v_\infty(z) \not\equiv 0 \tag{7.45}$$

To prove the claim, set

$$\widetilde{G}_k = G_k / \sigma_k^{2\alpha}$$

Using the elliptic theory of b-differential operators from section 6.2, we will prove that, for $\epsilon' < \epsilon$ as above and $0 < \gamma' < \gamma < 1$ we have the inequality

$$\|\widetilde{v}_k\|_{1+\epsilon,2,\gamma,1} \leq C \|\widetilde{v}_k\|_{1+\epsilon',2,\gamma',2} \,. \tag{7.46}$$

Note the shift in regularity and the fact that the norm on the right is on a ball of larger radius than the norm on the left. The L.H.S. is bounded from below by (7.41), so the right hand side is also bounded from below. By the compact containment (7.32), the v_{k,σ_k} converge strongly in the norm on the left, thus they converge to a non-zero function, i.e. (7.45) holds.

Thus it remains to prove the estimate (7.46). To do so, we apply Taylor's theorem to the Harmonic map operator τ around the map $id : (D, g_\alpha) \longrightarrow (D, \widetilde{G}_k)$. Let $r^c \dot{C}_b^{k,\gamma}(C_\alpha)$ denote the set of maps $v \in r^c C_b^{k,\gamma}(D(2))$ which vanish on $\partial D(2)$. By section 6.1 we have

$$\begin{aligned} \tau(w_k, g_\alpha, \widetilde{G}_k) &= \tau(id, g_\alpha, \widetilde{G}_k) + L_k \widetilde{v}_k + Q_k(\widetilde{v}_k) \\ L_k \widetilde{v}_k &= -Q_k(\widetilde{v}_k) \end{aligned}$$

All the materlial in section 6.2 applies to

$$L_k : r^{1+\epsilon} \dot{C}_b^{j,\gamma}(D(2)) \longrightarrow r^{1+\epsilon-2\mathfrak{a}} C_b^{j-2,\gamma}(D(2)). \tag{7.47}$$

In particular, it is Fredholm for any γ, j, and ϵ small, and it has an generalized inverse $\mathcal{G}_k$ which satisfies the mapping properties analogous to (6.23). By Lemma 6.6, (7.47) is injective, so $\mathcal{G}_k L_k = I$. From (6.9), for $\epsilon' < \epsilon$ as above, the inequality $\|Q_k(\widetilde{v}_k)\|_{1+2\epsilon'-2\alpha,j,\gamma,2} \leq C \|\widetilde{v}_k\|_{1+\epsilon',j+1,\gamma,2}$. Let $\chi(r)$ be a cutoff function that is 1 on the D and supported in $D(2)$. Then we have

$$\mathcal{G}_k \chi Q_k(\widetilde{v}_k) = -\mathcal{G}_k \chi L_k(\widetilde{v}_k) = -\mathcal{G}_k [L_k, \chi] \widetilde{v}_k - \chi \widetilde{v}_k,$$

so

$$\chi \widetilde{v}_k = \mathcal{G}_k \chi Q_k(\widetilde{v}_k) - \mathcal{G}_k [L_k, \chi] \widetilde{v}_k.$$

Since $[L_k, \chi]$ is the zero operator near the cone point, so satisfies

$$[L_k, \chi] : r^{1+\epsilon} C_b^{j,\gamma}(D(2)) \longrightarrow r^N C_b^{j-1,\gamma}(D(2))$$

for any $N > 0$. Tracing through all of the boundedness properties above, we get

$$\begin{aligned} \|\widetilde{v}_k\|_{1+2\epsilon',2,\gamma,1} &\leq \|\chi \widetilde{v}_k\|_{1+2\epsilon',2,\gamma,2} \\ &= \|\mathcal{G}_k \chi Q_k(\widetilde{v}_k)\|_{1+2\epsilon',2,\gamma,2} - \|\mathcal{G}_k [L_k, \chi] \widetilde{v}_k\|_{1+2\epsilon',2,\gamma,2} \\ &\leq \|\widetilde{v}_k\|_{1+\epsilon',1,\gamma,2} \end{aligned}$$

But ϵ' is an arbitrary positive number less than ϵ, so regardless of the ϵ present in (7.41), we can choose $\epsilon > \epsilon' > \epsilon/2$ and (7.46) is proven.

This completes the proof of Claim 7.11 and thus the proof of Proposition 7.8. $\square$

8. Cone angle π

We now discuss the case

$$\mathfrak{p}_{=\pi} \neq \varnothing,$$

which involves only minor modifications of the above arguments.

Let $id = u_0 : (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G)$ be energy minimizing and fix $p \in \mathfrak{p}_{=\pi}$. Let $\phi_1 : D \longrightarrow (\Sigma_{\mathfrak{p}}, g)$ and $\phi_2 : D \longrightarrow (\Sigma_{\mathfrak{p}}, G)$, so that the ϕ are conformal and $\phi_i(0) = p$. As usual, pick conformal coordinates z and w, resp.. The double cover $f : D \longrightarrow D$ with $f(z) = z^2$ can be used to pull back G to a metric $\overline{G} := f^*\phi_2^*G$ on D with (not necessarily smooth) cone angle 2π. The map $w = \phi_2 \circ u_0 \circ \phi_1^{-1}$ lifts to a harmonic map

$$\begin{tikzcd} D \arrow[r, "\widetilde{w}"] \arrow[d, "f"] & (D, \overline{G}) \arrow[d, "f"] \\ D \arrow[r, "w"] & (D, \phi_2^* G) \end{tikzcd} \tag{8.1}$$

Since $\overline{G}$ has cone angle 2π, by section 2.2, in conformal coordinates v we can write

$$\overline{G} = e^{2\mu} \left| d\widetilde{w} \right|^2 .$$

Generically, the metric $\overline{G}$ is not smooth near 0.

The reason we treat this case separately is that the form of these harmonic maps near $\mathfrak{p}_{=\pi}$ is different than Form 2.2. We have

Form 8.1 ($\mathfrak{p}_{=\pi} \neq \varnothing$)**.** *We say that $u : (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G)$ is in Form 8.1 (with respect to g and G) if*

(1) $u \in \mathrm{Diff}(\Sigma_{\mathfrak{p}})$
(2) $u \sim id$
(3) Near $p \in \mathfrak{p}_{=\pi}$, if w is defined as in (8.1), then

$$\widetilde{w}(\widetilde{z}) = az + b\overline{\widetilde{z}} + v(\widetilde{z})$$

where $a, b \in \mathbb{C}$, $a \neq 0$, and

$$v \in r^{1+\epsilon} C_b^{2,\gamma}(D(R)),$$

for some sufficiently small $\epsilon > 0$.

It is easy to check that Lemma 5.1 holds, i.e. that harmonic maps in Form 8.1 have Hopf differentials that are holomorphic with at worst simple poles at p. In fact, since $\Phi(\widetilde{w})$ is invariant under the deck transformation, we can just compute $\Phi(\widetilde{w})$ and use $\Phi(w) = f_*\Phi(\widetilde{w})$.

$$\begin{aligned} \Phi(\widetilde{w}) &= e^{2\mu} v_{\widetilde{z}} \overline{v}_{\widetilde{z}} \\ &= (a\overline{b} + \mathcal{O}(|\widetilde{z}|)) d\widetilde{z}^2. \end{aligned}$$

By holomorphicity and the invariance of Φ under the deck transformation,

$$\Phi(\widetilde{w}) = (a\overline{b} + g(\widetilde{z}^2)) d\widetilde{z}^2,$$

where g is a holomorphic function with $g(0) = 0$. Thus

$$\begin{aligned}\Phi(w) &= f_* \Phi(\widetilde{w}) \\ &= \frac{1}{4}\left(\frac{a\overline{b}}{z} + g(z)\right) dz^2.\end{aligned}$$

Thus, we have shown more than Lemma 5.1, namely,

Lemma 8.2. *Suppose $u : (\Sigma_{\mathfrak{p}}, g) \longrightarrow (\Sigma_{\mathfrak{p}}, G)$ is harmonic and is in Form 8.1 with $a \neq 0$, then u is actually in Form 2.2 if and only if $\Phi(u)$ extends holomorphically over $\mathfrak{p}_{=\pi}$.*

it follows trivially that *all of the results of section 5 hold*, since the only way Form 2.2 came into the picture was in proving Lemma 5.1.

Given a harmonic u_0 in Form 8.1, the space $\mathcal{B}^{1+\epsilon}(u_0)$ is defined so that $u \in \mathcal{B}^{1+\epsilon}(u_0)$ if and only if near $q \in \mathfrak{p}$, $u - u_0 \in r^{1+\epsilon} C_b^{2,\gamma}$. For $p \in \mathfrak{p}_{=\pi}$, writing the lift of w_0 as

$$\widetilde{w}_0(\widetilde{z}) = a_0 \widetilde{z} + b_0 \overline{\widetilde{z}} + v_0(\widetilde{z})$$

we see that

$$u \in \mathcal{B}^{1+\epsilon}(u_0) \implies \widetilde{w}(z) = a_0 \widetilde{z} + b_0 \overline{\widetilde{z}} + v(\widetilde{z}) \text{ for some } v \in r^{1+\epsilon} C_b^{2,\gamma}$$

where w is the localized lift of u from (8.1). As above, we allow the tension field operator τ to act on a space of geometric perturbations. Near $p \in \mathfrak{p}_{=\pi}$ they can be described simply; given $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, consider the (locally defined) map

$$\widetilde{w}(\widetilde{z}) = (a_0 + \lambda_1)\widetilde{z} + (b_0 + \lambda_2)\overline{\widetilde{z}} + v(\widetilde{z}). \tag{8.2}$$

Following the above arguments we let $\mathcal{D}'$ be a space of automorphisms which look like (3.10) near $q \in \mathfrak{p} - \mathfrak{p}_{=\pi}$ and which lift to look like (8.2) near each $p \in \mathfrak{p}_{=\pi}$.

8.1. **$\mathcal{H}(\mathfrak{q})$ is open ($\mathfrak{p}_{=\pi} \neq \varnothing$).** We state and sketch the proof of the main lemma

Lemma 8.3. *The tension field operator τ acting on $\mathcal{B}^{1+\epsilon}(u_0) \circ \mathcal{D}' \circ \mathcal{T}_{>\pi}$ is C^1. Its linearization $L = D_u \tau$ is bounded as a map*

$$L : r^{1+\epsilon} \mathcal{X}_b^{2,\gamma} \oplus T_{id}\mathcal{D}' \oplus T_{id}\mathcal{T}_{>\pi} \longrightarrow r^{1+\epsilon-2\alpha} \mathcal{X}_b^{0,\gamma},$$

and is transverse to $(T\mathcal{C}onf_0 \cap r^{1+\epsilon-2\mathfrak{a}} \mathcal{X}_b^{0,\gamma})^{\perp}$.

By Remark 6.11, the lemma implies the openness statement exactly as it did in the case $\mathfrak{p}_{=\pi} = \varnothing$.

The map

$$L : r^{1+\epsilon} \mathcal{X}_b^{2,\gamma} \longrightarrow r^{1+\epsilon-2\alpha} \mathcal{X}_b^{0,\gamma} \tag{8.3}$$

is Fredholm for small ϵ, as is $L : r^{1-\epsilon} \mathcal{X}_b^{2,\gamma} \longrightarrow r^{1-\epsilon-2\alpha} \mathcal{X}_b^{2,\gamma}$. The latter map has cokernel $\mathcal{K} = \operatorname{Ker} L|_{r^{1+\epsilon-2\mathfrak{a}} \mathcal{X}_b^{2,\gamma}}$. The cokernel of (8.3) can again be written as $\widetilde{W} \oplus \mathcal{K}$ where $\widetilde{W}$ consists of vectors $\psi \in r^{1-\epsilon} \mathcal{X}$ with $L\psi \in r^{1+\epsilon-2\alpha} \mathcal{X}$, and again such vectors have expansions determined by the indicial roots. Near $p \in \mathfrak{p}_{=\pi}$, using the lifts at the beginning of this section makes the asymptotics easy to calculate. Note that

$$H(u, g, G) = (\phi_1)_* f_* H(\widetilde{w}, D, \overline{G}),$$

where $\widetilde{z}$ is defined by (8.1), which immediately implies (initially on $\mathcal{X}^\infty$), that near $p \in \mathfrak{p}_{=\pi}$,

$$L_{u_0,g,G} = (\phi_1)_* f_* D_{\widetilde{w}} H|_{\widetilde{w}_0,D,\overline{G}}$$

If $\overline{L} := D_{\widetilde{z}} H|_{\widetilde{w}_0,D,\overline{G}}$, then setting $\widetilde{\overline{L}} = (|\widetilde{z}|^2/4)\overline{L}$ we have

$$\widetilde{\overline{L}} = (\widetilde{z}\partial_{\widetilde{z}})\left(\overline{\widetilde{z}}\partial_{\overline{\widetilde{z}}}\right) + E(\widetilde{z}).$$

Any $\psi \in r^{1+\epsilon-2\alpha}\mathcal{X}$ is in $r^\epsilon C_b^{2,\gamma}$ near $p \in \mathfrak{p}_{=\pi}$ since $\alpha_p = 1/2$. It is now easy to see that a solution $L\psi = 0$ has lift

$$\begin{aligned} \overline{\psi}(\widetilde{z}) &= \lambda_1 \widetilde{z} + \lambda_2 \overline{\widetilde{z}} + \overline{\psi}' \\ \overline{\psi}' &\in r^{1+\epsilon} C_b^{2,\gamma} \end{aligned} \tag{8.4}$$

near such a p, and this shows that

$$L\left(r^{1+\epsilon}\mathcal{X}_b^{2,\gamma} \oplus \mathcal{D}'\right) \oplus \mathcal{K} = r^{1+\epsilon-2\mathfrak{a}}\mathcal{X}_b^{0,\gamma}$$

In analogy with Lemma 6.8, we have that, near $p \in \mathfrak{p}_{=\pi}$, elements of $\mathcal{K}$ look like (8.4). All of the material in section 6 follows after replacing $\mathcal{D}$ by $\mathcal{D}'$.

8.2. **$\mathcal{H}(\mathfrak{q})$ is closed ($\mathfrak{p}_{=\pi} \neq \varnothing$).** Since by Lemma 8.2 solutions to (HME($\mathfrak{q}$)) are in Form 2.2, the proof the $\mathcal{H}_2$ is closed is identical to the proof in the $\mathfrak{p}_{=\pi} = \varnothing$ case.

For (HME($\mathfrak{q}$)), the proof again requires only minor modifications. The main difference is the following; consider a sequence of maps u_k and converging metrics metrics $G_k \to G_0$ and $g_k \to g_0$ as in the statement of Theorem 7.1. In the same way as in the $\mathfrak{p}_{=\pi} = \varnothing$ case, we reduce to the local analysis of the u_k near a cone point $p \in \mathfrak{p}_{=\pi}$. Here we assume that the maps are in Form 8.1 with

$$|a_k| > |b_k|. \tag{8.5}$$

By analogy with the treatment of the λ_k in the $\mathfrak{p}_{=\pi} = \varnothing$ case, we want want to show that the condition (8.5) persists in the limit, and for this we need some uniform control of the a_k, b_k. In fact, we claim that

$$|a_k| \leq c \text{ and } |a_k| - |b_k| \geq c > 0,$$

for some uniform constant c. As in the previous case, the a_k and b_k are relate to the energy density e_k and the function h_k defined in (7.8). We have the following

$$\begin{aligned} h_k(z) &\sim |a_k|^{2\alpha}\left|1 + \frac{b_k}{a_k}e^{-2i\theta}\right|^{2(\alpha-1)} + f_2 \\ e_k(z) &\sim |a_k|^{2(\alpha-1)}\left|1 + \frac{b_k}{a_k}e^{-2i\theta}\right|^{2(\alpha-1)} (|a_k|^2 + |b_k|^2) + f_1 \\ f_i &\in r^\epsilon C_b^{2,\gamma} \end{aligned} \tag{8.6}$$

If follows that the e_k are uniformly bounded, since they are still subsolutions. In the second line, choosing θ so that $\frac{b_k}{a_k}e^{-2i\theta} = \left|\frac{b_k}{a_k}\right| < 1$ and using the uniform bound $e_k < c$ gives $|a_k| < c$. We apply Lemma 7.6 to the $\log h_k/\delta_k$ for δ_k defined as in (7.23), noting that the hypotheses are satisfied by (8.6). As above, this leads to the lower bound

$$\inf_{z \in D-0} h_k(z) \geq c > 0,$$

so choosing θ such that

$$1 + \frac{b_k}{a_k} e^{-2i\theta} = 1 - \left| \frac{b_k}{a_k} \right|$$

we get that

$$|a_k| - |b_k| \geq c > 0$$

The rest of the argument proceeds as in the $\mathfrak{p}_{=\pi} = \varnothing$ case, with a_k playing the role of λ_k. Assuming the same type of blow-up near $p \in \mathfrak{p}_{=\pi}$, and rescaling in the exact same way, on the local double cover a harmonic map of $w_\infty : \mathbb{C} \longrightarrow \mathbb{C}$ results with $w_\infty = a_\infty z + b_\infty \overline{z} + v_\infty$ with $v_\infty \in r^{1+\epsilon} C_b^{2,\gamma}$ not identically zero. This is a contradiction by the following the argument from [D], which we outline briefly; an orientation preserving harmonic mapping map of $\mathbb{C}$ can be written, globally, as a sum $f + \overline{g}$ where f and g are holomorphic. The ratio $\partial_z g / \partial_z f$ is bounded by the orientation preserving property and is clearly holomorphic, hence constant. Integrating proves the statement.

9. H is continuously differentiable

We now discuss in detail the map (3.15). To study its properties we trivialize the bundle $\mathsf{E} \longrightarrow \mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0) \circ \mathcal{C} \times \mathcal{M}^*_{2,\gamma,\nu}(g_0, \mathfrak{p}, \mathfrak{a}) \times \mathcal{M}^*_{2,\gamma,\nu}(G_0, \mathfrak{p}, \mathfrak{a})$, and use the trivializing map to define the topology of E. We define a map

$$\Xi : \mathsf{E} \longrightarrow \mathcal{X}^{1+\epsilon-2\mathfrak{a}}_{0,\gamma}(u_0) \tag{9.1}$$

as follows. Let $((u \circ C, g, G), \psi) \in \mathsf{E}$ where $\psi \in \mathcal{X}^{1+\epsilon-2\mathfrak{a}}_{0,\gamma}(u \circ C)$. By the definition of $u \in \mathcal{B}^{1+\epsilon}(u_0)$, there is a unique $\widetilde{\psi} \in \mathcal{X}^{1+\epsilon}(u_0)$ so that

$$u = \exp_{u_0}(\widetilde{\psi})$$

Assuming for the moment that $C = id$, let $\Xi((u, g, G), \psi) = \Xi_u(\psi)$ where

$$\Xi_u(\psi) \;\; := \;\; \begin{array}{c} \text{parallel translation of } \psi \text{ along} \\ \gamma_t := \exp_{u_0}(t\widetilde{\psi}) \text{ from } t = 1 \text{ to } t = 0. \end{array} \tag{9.2}$$

In general, motivated by pointwise conformal invariance of τ (see (3.6)), define $\Xi((u \circ C, g, G), \psi) = \Xi_u(\psi \circ C^{-1})$. Obviously, Ξ is an isomorphism on each fiber, and we endow E with the pullback topology induced by Σ. Thus a section σ of E is C^1 if and only if $\Xi \circ \sigma$ is C^1. The purpose of this section is to prove Proposition 3.4, which states that if g_0 and G_0 satisfy Assumption 3.1, then the map 9.1 is C^1.

We reduce the proposition to a computation in local coordinates. Near $p \in \mathfrak{p} - \mathfrak{p}_{=\pi}$, we heave

$$\begin{aligned} H(u \circ C, g, G) &= \Xi\left(\tau(u, C^* g, G)\right) \\ &= \Xi_u\left(\tau^i(u, C^* g, G)\partial_i\right) \\ &= \tau^i(u, C^* g, G)(\Xi_u)_i^{\ j}\partial_j, \end{aligned}$$

where $(\Xi_u)_i^{\ j}$ is the local coordinate expression for parallel translation of ∂_α along the path in (9.2).

Since (u_0, g_0, G_0) solves (HME($\mathfrak{q}$)) we have $\tau(u_0, g_0, G_0) = 0$, and since G and G_0 have the same the same conformal coordinates near $p \in \mathfrak{p}$, we have

$$\begin{aligned} G_0 &= \rho_0(u)\,|du|^2 = c_0 e^{2\mu_0}\,|u|^{2(\alpha-1)}\,|du|^2 \\ G &= \rho(u)\,|du|^2 = c e^{2\mu}\,|u|^{2(\alpha-1)}\,|du|^2 \end{aligned} \tag{9.3}$$

and

$$\begin{aligned} g_0 &= \sigma_0(z)\,|dz|^2 = c_0 e^{2\lambda_0}\,|z|^{2(\alpha-1)}\,|dz|^2 \\ g &= \sigma(z)\,|dz|^2 = c e^{2\lambda}\,|z|^{2(\alpha-1)}\,|dz|^2 . \end{aligned}$$

Recall that, in local coordinates

$$\tau(u, g, G)(z) = \tau(u, g, G) = \frac{4}{\sigma}\left(u_{z\overline{z}} + \frac{\partial \log \rho}{\partial u} u_z u_{\overline{z}}\right)$$

By (9.3),

$$\frac{\partial \log \rho(u_0)}{\partial u} = 2\frac{\partial \mu}{\partial u} + \frac{\alpha - 1}{u_0}.$$

Thus we have

$$\tau^i(u, C^* g, G) \in \frac{4}{\sigma} r^{\epsilon-1} C_b^{0,\gamma} = r^{1+\epsilon-2\mathfrak{a}} C_b^{0,g} \tag{9.4}$$

Near $p \in \mathfrak{p}_{=\pi}$, recall that $C = D'_{\lambda_1,\lambda_2} \circ T_w$ where $T_w(z) = z - w$ and $D'_{\lambda_1,\lambda_2}(\widetilde{z}) = \lambda_1 \widetilde{z} + \lambda_2 \overline{\widetilde{z}}$, for $\widetilde{z}^2 = z$ and $\widetilde{u}^2 = u$ coordinates on the local double cover. Again we have, locally

$$\begin{aligned} H(u \circ D_{\lambda_1,\lambda_2} \circ T_w, g, G) &= \Xi\left(\tau(u \circ D_{\lambda_1,\lambda_2}, T_w^* g, G)\right) \\ &= \Xi_u\left(\tau^i(u \circ D_{\lambda_1,\lambda_2}, T_w^* g, G)\partial_i\right) \\ &= \tau^i(u \circ D_{\lambda_1,\lambda_2}, T_w^* g, G)(\Xi_u)_i{}^j \partial_j, \end{aligned}$$

The local computation of τ can now be done in the lifted coordinates $\widetilde{z}$, where $\widetilde{u} = a\widetilde{z} + b\overline{\widetilde{z}} + v$ for $v \in r^{1+\epsilon} C_b^{2,\gamma}$ coordinates. The pulled back tension field is

$$\frac{4}{\sigma}\left(\widetilde{u}_{\widetilde{z}\overline{\widetilde{z}}} + \frac{\partial \log \rho}{\partial \widetilde{u}}((a + \lambda_1) + \widetilde{u}_{\widetilde{z}})((b + \lambda_2) + \widetilde{u}_{\overline{\widetilde{z}}})\right)$$

So by $\frac{\partial \log \rho(\widetilde{u}_0)}{\partial \widetilde{u}} = 2\frac{\partial \widetilde{\mu}}{\partial \widetilde{u}}$, we have $\tau^i \in r^{\epsilon} C_b^{0,\gamma}$, which is (9.4) in this context.

As for the expression $(\Xi_u)_i{}^j \partial_j$, a simple exercise in ODEs shows that (if we assume $u - z \in r^{1+\epsilon} C_b^{2,\gamma}$), then $(\Xi_u)_i{}^j \partial_j - \partial_i \in r^{1+\epsilon} C_b^{0,\gamma}$. This immediately implies

Thus we have established

Lemma 9.1. *If (u_0, g_0, G_0) satisfy (HME($\mathfrak{q}$)) and u_0 is in Form 2.2, then*

$$\begin{aligned} \tau : \mathcal{B}^{1+\epsilon}_{2,\gamma}(u_0) \circ \mathcal{D} \times \mathcal{M}^*_{2,\gamma,\nu}(G_0, \mathfrak{p}, \mathfrak{a}) &\longrightarrow \mathsf{E} \\ (u \circ C, g, G) &\longrightarrow \tau(u \circ C, g, G) \end{aligned}$$

is C^1 near u_0.

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